THE FOREMOST TEXTBOOK OF MODERN TIMES*

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The most influential mathematics textbook of ancient times (or, for that matter, of all times) is easily named. The *Elements* of Euclid, appearing in over a thousand editions [1], has set the pattern for the teaching of elementary geometry ever since it was composed more than two and a quarter millenia ago. The medieval textbook which most strongly influenced mathematical development is not so easily selected; but a good case can be made out for *Al jabr wa'l muquabala* of Al-Khowarizmi [2], just about half as old as the *Elements*. From this Arabic work algebra took its name and, to a great extent, its origin. Is it possible to indicate a modern textbook of comparable influence and prestige? Some would mention the *Géométrie* of Descartes or the *Principia* of Newton or the *Disquisitiones* of Gauss; but in pedagogical significance these classics fell far short of a work less well known. The *Géométrie* was not strictly a textbook, and hence most mathematicians learned their analytic geometry from the works of other authors, such as Schooten, De Witt, Sluze, and Lahire. The *Principia*, the greatest of all works in the field of science, affected the course of pure mathematics only indirectly; few readers understood the elements of the calculus which it contained, and the effective teachers of the differential calculus were Leibniz, L'Hospital, and the Bernoullis. The *Disquisitiones*, a work of great profundity, was too specialized to make its influence widely felt except among ardent number-theorists. It is perhaps significant that none of these three modern works appeared in what may be considered the greatest age of textbooks in recent times. The eighteenth century often is characterized as a prosy age in the history of mathematics, for it contributed no single discovery which captured the imagination as had analytic geometry and the calculus. And yet the century was of capital importance in the consolidation of earlier work, a task which was facilitated by the appearance of outstanding textbook writers. At the opening of the century one finds the texts of L'Hospital dominating the fields of analytical conics and the calculus; at the close there were the textbooks of Lacroix which covered the whole elementary field and which appeared in dozens of editions [3], not to mention the Legendre *Euclid* of the same time. But over these well known textbooks there towers another, a work which appeared in the very middle of the great textbook age and to which virtually all later writers admitted indebtedness. This was the *Introductio in analysin infinitorum* of Euler, published in two volumes in 1748 [4]. Here in effect Euler accomplished for analysis what Euclid and Al-Khowarizmi had done for synthetic geometry and elementary algebra, respectively. The function concept and infinite processes had arisen by the seventeenth century, yet it was Euler's *Introductio* which fashioned these into the third member of the mathematical triumvirate comprising geometry, algebra, and analysis. From the point

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of view of leading textbooks, then, one might refer (with, of course, some oversimplification) to geometry as ancient, algebra as medieval, and analysis as modern.

Euler was not the first to use the word "analysis," or even to incorporate it into the title of a book; but he did give the word a new emphasis. Plato's analysis had reference to the logical order of steps in geometrical reasoning, and the analytic art of Viète was akin to our algebra; but the analysis of Euler comes close to the modern orthodox discipline, the study of functions by means of infinite processes, especially through infinite series. It is in this newer sense that Euler, especially after 1748, used the word; and it is for this reason that he has been referred to as "analysis incarnate" [5]. The word, analysis, took on a new lease of life. Euler himself used it in the titles of dozens of his published papers; and soon others were publishing books on "analytic optics" and "analytical mechanics," on "analytical trigonometry" and "analytic geometry." Euler avoided the phrase "analytic geometry," probably to obviate confusion with the older Platonic usage; yet the second volume of the *Introductio* has been referred to, not inappropriately, as "the first text on analytic geometry" [6], and this appeared more than a century after the publication of *La géométrie*! It is the earliest systematic graphical study of functions, not only of a single variable, but of two as well; and Euler's analysis is transcendental as well as algebraic. The notion of "elementary function" stems largely from the *Introductio*, but the book contains also such higher curves as $y=x^{\sqrt{3}}$ and $y^x=x^y$. Polar coordinates had appeared in at least half a dozen earlier works, including Newton's *Method of fluxions*; but the clarity and generality of Euler's treatment in the *Introductio* were such that most subsequent writers traced the use of polar coordinates back to this book. Here, in fact, the spiral of Archimedes appears in its dual form, probably for the first time. The use of parametric equations, implicit even in the work of Descartes, was first systemized in the *Introductio*; and here also one finds formalized for the first time the equations for the transformation of coordinates for two and three dimensions, the latter in a form still referred to as "Euler's equations." The *Introductio* was the first textbook to recognize the five proper general quadric surfaces as members of a single family, a century and a half after Kepler had done the same for the conics, and the names proposed were very similar to those now adopted [7]. In this same book Euler also did for plane quartic curves what Newton had done for cubics—he ordered them according to genus and species.

The word "first" is a hazardous one to use in the history of mathematics, and yet it has been applied freely to Euler's *Introductio*. The cases already cited by no means exhaust the respects in which this textbook was first. It contains the earliest algorithmic treatment of logarithms as exponents and of the trigonometric functions as numerical ratios. It was the first textbook to list systematically the multiple-angle formulas, calling attention to the periodicities of the functions; and it included the first general analytic treatment of these as infinite products, as well as their expansion into infinite series. The well-

known "Euler identities," relating the trigonometric functions to imaginary exponentials, are also found here. The first volume contains as well an exposition of continued fractions and some excellent work of the zeta function and number theory [8].

In scope alone the *Introductio* ranks among the greatest of textbooks, for it is doubtful that any other essentially didactic work includes as large a portion of original material which survives in the college courses of today. Yet the book is outstanding also for its pedagogical lucidity. The immortal Gauss, a man not given to exaggerated expressions of flattery, held that "The study of Euler's works will remain the best school for the various fields of mathematics, and nothing can replace it" [9]. Written as it was more than two hundred years ago (a letter from Euler to D'Alembert indicates that it was completed by 1745), the *Introductio in analysin infinitorum* nevertheless can be read with comparative ease by the modern student—unlike the *Géométrie*, the *Principia*, or the *Disquisitiones*. Not only is the viewpoint quite similar to that of today; even the terminology and notation are almost modern—or perhaps, as Struik has well written, "we should better say that our notation is almost Euler!" [10]. Under the circumstances one should expect that a textbook exhibiting the qualities of the *Introductio* would boast an impressive list of editions and translations; but the facts belie this. The work was not reprinted until the time of the author's death, thirty-five years later; and, including reprintings and incomplete translations, the fewer than a dozen editions are about equally distributed among the three languages Latin, French, and German [11]. No English translation has appeared, and a partial Russian translation apparently has not been published. However, that the worth of a textbook is not necessarily measured by the number of its editions is conclusively evidenced by the *Introductio*, the influence of which was unusually pervasive. Almost without exception the authors of the ubiquitous compendia of the second half of the eighteenth century refer to Euler as the source of their analysis. The *Introductio* became, in a sense, the prototype of modern textbooks. Is not imitation the sincerest form of flattery?

References

1. An excellent English edition is *The thirteen books of the Elements*, translated from the text of Heiberg with introduction and commentary by T. L. Heath, 3 vols., Cambridge, 1908.

2. See *Robert of Chester's Latin translation of the Algebra of Al-Khowarizmi*, with an introduction, critical notes, and an English version by L. C. Karpinski, New York, 1915.

3. In 1848, for example, there appeared at Paris the 20th edition of his *Traité élémentaire d'arithmétique* and the 16th edition of his *Éléments de géométrie*; and ten years later his *Éléments d'algèbre* appeared in a 20th edition.

4. Published at Lausanne.

5. See, e.g., E. T. Bell, *Men of mathematics* (New York, 1937), chapter 9.

6. D. J. Struik, *A concise history of mathematics* (2 vols., New York, 1948), II, 169.

7. His names were elliptoides, elliptico-hyperbolicae, hyperbolico-hyperbolicae, elliptico-parabolicae, and parabolico-hyperbolicae.

8. An extensive account of this work is found in the preface, by Andreas Speiser, to series 1, volume 9, of Euler's *Opera omnia* (Geneva, 1945).

9. *Ibid.*, p. viii. Speiser adds to the words of Gauss: "The *Introductio* in this connection may stand in first place."

10. Struik, *op. cit.*, II, 174.

11. Latin editions appeared in 1748, 1783, 1797, and in volumes 9 and 10 of Euler's *Opera*, series 1; French editions were published in 1785, 1796, and 1835; German editions appeared in 1788, 1835, and 1885. For bibliographic details on Euler's multitudinous works see Gustav Eneström, "Verzeichnis der Schriften Leonhard Eulers," *Jahresbericht der Deutschen Mathematiker-Vereinigung, Ergänzungsband* IV, Leipzig, 1910.. Cf. also P. H. Fuss, *Correspondance mathématique et physique de quelques célèbres géomètres du XVIII^ème siècle. Précédée d'une notice sur les travaux de Léonard Euler* (2 vols., St. Pétersbourg, 1843); and Felix Müller, "Über bahnbrechende Arbeiten Leonhard Eulers aus der reinen Mathematik," *Abhandlungen zur Geschichte der mathematischen Wissenschaften*. XXV (1907), 61–116.

PHYSICAL FAMILIES IN THE GRAVITATIONAL FIELD OF FORCE

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1. Introduction. The important physical families of curves connected with an arbitrary positional field of force are: (a) dynamical trajectories, the paths of motion of a particle; (b) general catenaries, the curves formed from the resulting equilibrium when an inextensible flexible string is suspended in any arbitrary field of force; (c) brachistochrones, the curves in a general conservative field of force along which the time of the constrained motion between any two points is least; (d) velocity curves, the curves in an arbitrary field of force, to any one of which there corresponds a speed v_0 such that a particle, starting from any lineal element of the curve with the speed v_0, describes a trajectory osculating the curve.

These four systems of physical curves may all be obtained as special cases of the following general problem [1]: To find all curves along which a constrained motion is possible such that the pressure P, normal to the curve, is proportional to the normal component N of the force vector. We shall term any such family a *physical system* S_k. Thus along any curve of a system S_k, we have $P=kN$, where $k(\neq -1)$ is the constant proportionality.

The important special cases of physical interest of a system S_k are (a) dynamical trajectories, $k=0$, (b) general catenaries, $k=1$; (c) generalized brachistochrones, $k=-2$: (d) velocity curves, $k=\infty$. These four families are systems S_0, S_1, S_{-2}, S_∞.

In our previous work [2], we have discussed systems S_k of ∞^3 curves in any given arbitrary field of force. It is our purpose to investigate the system S_k in the gravitational field of force where the force vector is identically constant. The integration of a system S_k can be obtained in terms of a single quadrature. If k is a rational, then the only systems S_k which may be expressible

in terms of elementary functions are those for which k is of the form $k=(1-n)/n$, or of the form $k=(1-2n)/(1+2n)$, where n is an integer, positive, negative, or zero. If it be required that a system S_k be expressible in terms of elementary functions and that k be an integer, then the only solutions are (a) dynamical trajectories S_0, (b) catenaries S_1, (c) brachistochrones S_{-2}, (d) velocity curves S_∞, and (e) another system S_{-3} of circles.

The development of the present article will be from elementary principles.

2. Preliminary formulas. The time t is measured in seconds and the magnitude of the force vector F is measured in dynes. The distance s and the mass m may be measured in centimeters and grams, or in feet and pounds, respectively. The force exerted by m grams or m pounds is mg dynes where g is 980 or 32 according as the first or second system of units is used [3].

For the two systems of units discussed above, Newton's second law of motion is $F=ma$, where F and a are the force and acceleration vectors, and m is the mass.

Let a particle of mass m traverse a curve C. At a time t_0, let its position on C be (x_0, y_0), its speed be v_0, and the inclination of its direction of motion be θ_0. At the time t_0, the horizontal and vertical components of its velocity vector are $(v_0 \cos\theta_0, v_0 \sin\theta_0)$.

The normal component a_n and the tangential component a_t of the acceleration vector a of the particle are

$$a_n = \frac{v^2}{r}, \qquad a_t = \frac{dv}{dt} = v\frac{dv}{ds}, \tag{1}$$

where v is the speed, s is the length of arc, and r is the radius of curvature.

The rectangular components of the force vector F in a gravitational field of force, acting on a particle of mass m, are

$$\phi(x, y) = 0, \qquad \psi(x, y) = -mg. \tag{2}$$

If the particle is constrained to move along a given curve C, the normal component N and the tangential component T of the force vector are

$$N = \psi\frac{dx}{ds} - \phi\frac{dy}{ds} = -mg\frac{dx}{ds}, \qquad T = \phi\frac{dx}{ds} + \psi\frac{dy}{ds} = -mg\frac{dy}{ds}. \tag{3}$$

By (1) and (3), the pressure P of the particle along this curve is given by the formula

$$P = \frac{mv^2}{r} - N = m\left(\frac{v^2}{r} + g\frac{dx}{ds}\right). \tag{4}$$

Also by (1) and (3), the speed v of the particle obeys the condition $vdv/ds = -gdy/ds$. Integrating this, the result is the energy equation

$$\frac{v^2}{2} = h - gy, \tag{5}$$

where the energy constant has the value

$$h = \frac{v_0^2}{2} + gy_0. \tag{6}$$

3. The systems S_k. A system S_k, where $k \neq -1$, in an arbitrary field of force is defined to consist of curves along each of which a constrained motion is possible such that the pressure $P = mv^2/r - N$ is proportional to the normal component N of the force vector F. Thus $P = kN$ where $k \neq -1$.

The equations of motion of a system S_k in a general field of force are

$$mv^2 = (k+1)rN, \qquad mv\frac{dv}{ds} = T. \tag{7}$$

For each fixed k, a system S_k is a three parameter family of curves. Thus S_k consists of ∞^3 curves.

By (3) and (4), the equations of motion of a system S_k of ∞^3 curves in the gravitational field of force are

$$v^2 = -(k+1)gr\frac{dx}{ds}, \qquad v\frac{dv}{ds} = -g\frac{dy}{ds}. \tag{8}$$

By use of Euler's equation in the Calculus of Variations, it can be shown that these are the extremals of the variation problem

$$\int v^{k+1}ds = \text{min.}, \quad \text{or} \quad \int (h - gy)^{(k+1)/2}ds = \text{min.} \tag{9}$$

We consider the *system S_∞* of velocity curves. In an arbitrary field of force, a curve C is a velocity curve corresponding to the speed v_0 if any dynamical trajectory, starting from any lineal element of C with the speed v_0, is initially osculated by C. It is found that the system of velocity curves is the limiting case of a system S_k as k becomes infinite. For this reason, we denote a velocity system by the symbol S_∞.

In a general field of force, the differential equation of a velocity system S_∞ is

$$Nr = mv_0^2. \tag{10}$$

A system S_∞ consists of ∞^3 curves.

In a gravitational field of force, the system S_∞ of velocity curves is given by the differential equation

$$gr\frac{dx}{ds} = -v_0^2. \tag{11}$$

It can be shown by Euler's equation in the Calculus of Variations that these are the extremals of the variation problem

$$\int e^{-gy/v_0^2} ds = \min. \tag{12}$$

4. The integration of the velocity system S_∞. Let θ denote the inclination of the tangent line to a curve. Equation (11) can be written in the form

$$g dx = -v_0^2 d\theta, \tag{13}$$

where $dy = \tan\theta dx$.

The velocity system S_∞ in the gravitational field of force is

$$x = x_0 - \frac{v_0^2}{g}(\theta - \theta_0), \qquad y = y_0 + \frac{v_0^2}{g}\log\frac{\cos\theta}{\cos\theta_0}, \tag{14}$$

where θ is the inclination of the tangent line. The explicit form of any velocity curve is

$$y = y_0 + \frac{v_0^2}{g}\log\left[\cos\frac{g}{v_0^2}(x - x_0) + \tan\theta_0 \sin\frac{g}{v_0^2}(x - x_0)\right]. \tag{15}$$

5. The integration of the system S_k where $k \neq -1$ and $k \neq \infty$. By (5) and (8), we find

$$v^2 = 2(h - gy) = -(k+1)gr\frac{dx}{ds}. \tag{16}$$

Thus it is necessary to solve the equations

$$dx = \frac{2(gy - h)}{(k+1)g}d\theta, \qquad dy = \tan\theta dx. \tag{17}$$

From (17), it is deduced that

$$\frac{-gdy}{h - gy} = \frac{2}{k+1}\tan\theta d\theta. \tag{18}$$

Using (6), it is found that

$$y = y_0 + \frac{v_0^2}{2g}\left[1 - \left(\frac{\cos\theta_0}{\cos\theta}\right)^{2/(k+1)}\right]. \tag{19}$$

Also

$$dx = \cot\theta dy = -\frac{v_0^2}{g(k+1)}(\cos\theta_0)^{2/(k+1)}(\sec\theta)^{2/(k+1)}d\theta. \tag{20}$$

Thus

$$x = x_0 - \frac{v_0^2}{g(k+1)}(\cos\theta_0)^{2/(k+1)}\int_{\theta_0}^{\theta}(\sec\theta)^{2/(k+1)}d\theta. \tag{21}$$

Any system S_k where $k \neq -1$ and $k \neq \infty$ in the gravitational field of force consists of the ∞^3 curves

$$\begin{aligned} x &= x_0 - \frac{v_0^2}{g(k+1)}(\cos\theta_0)^{2/(k+1)}\int_{\theta_0}^{\theta}(\sec\theta)^{2/(k+1)}d\theta, \\ y &= y_0 + \frac{v_0^2}{2g}\left[1 - \left(\frac{\cos\theta_0}{\cos\theta}\right)^{2/(k+1)}\right], \end{aligned} \tag{22}$$

where θ is the inclination of the tangent line. In explicit form, this family of ∞^3 curves is given by the Tchebycheff integral

$$x = x_0 + \int_{y_0}^{y}\left[\sec^2\theta_0\left\{1 - \frac{2g}{v_0^2}(y - y_0)\right\}^{k+1} - 1\right]^{-1/2}dy. \tag{23}$$

By a theorem of Tchebycheff [4] this integral (23) is expressible in terms of elementary functions only in the case where $1/(1+k)$, or $(1-k)/2(1+k)$, is an integer. Hence $1/(1+k)=n$, or $(1-k)/2(1+k)=n$, where n is an integer. Therefore $k=(1-n)/n$, or $k=(1-2n)/(1+2n)$.

If k is rational, then the Tchebycheff integral (23) *is expressible in terms of elementary functions only in the cases $k=(1-n)/n$, or $k=(1-2n)/(1+2n)$, where n is an integer.*

The following result may be readily established:

The only systems S_k where $k(\neq -1)$ is an integer and which are expressible in terms of elementary functions are (a) S_0, *dynamical trajectories*; (b) S_1, *catenaries*; (c) S_{-2}, *brachistochrones* (*ordinary cycloids in this case*); (d) S_∞, *velocity curves*; (e) S_{-3}, *circles.*

The proof of the above result is as follows. In the first case, $1/(1+k)=n$ is an integer only when $k=0$ or $k=-2$. These give the systems S_0 of dynamical trajectories and S_{-2} of brachistochrones.

In the second case, it is desired to solve the equation

$$\frac{1-k}{2(1+k)} = n, \tag{24}$$

for integral values of k and n. Evidently k is odd, say $k=2m-1$. Then $n=(1-m)/2m$. Thus m is an odd integer such that $1-m$ is divisible by m. This can happen only when $m=1$ or $m=-1$. Hence $n=0$ or $n=-1$. Therefore $k=1$ or $k=-3$. The first solution gives S_1, the system of catenaries, and the second solution gives S_{-3}, a system which we now examine.

Substitute $k=-3$ into (22). The parametric equations of S_{-3} are

$$\begin{aligned} x &= x_0 - \frac{v_0^2}{2g}\tan\theta_0 + \frac{v_0^2}{2g\cos\theta_0}\sin\theta, \\ y &= y_0 + \frac{v_0^2}{2g} - \frac{v_0^2}{2g\cos\theta_0}\cos\theta. \end{aligned} \tag{25}$$

These are circles. The center (X, Y) and the radius R of any such circle are

$$X = x_0 - \frac{v_0^2}{2g}\tan\theta_0, \qquad Y = y_0 + \frac{v_0^2}{2g}, \qquad R = \frac{v_0^2}{2g\cos\theta_0}. \tag{26}$$

Substitute $k=0$ into (22). The system S_0 of dynamical trajectories is

$$\begin{aligned} x &= x_0 + \frac{v_0^2}{g}\sin\theta_0\cos\theta_0 - \frac{v_0^2}{g}\cos^2\theta_0\tan\theta, \\ y &= y_0 + \frac{v_0^2}{2g} - \frac{v_0^2}{2g}\cos^2\theta_0\sec^2\theta. \end{aligned} \tag{27}$$

These are the vertical parabolas

$$y = y_0 + \frac{v_0^2}{2g}\sin^2\theta_0 - \frac{g}{2v_0^2\cos^2\theta_0}\left(x - x_0 - \frac{v_0^2}{g}\sin\theta_0\cos\theta_0\right)^2. \tag{28}$$

Substitute $k=1$ into (22). The system S_1 of catenaries is

$$\begin{aligned} x &= x_0 - \frac{v_0^2}{2g}\cos\theta_0\log\frac{\sec\theta+\tan\theta}{\sec\theta_0+\tan\theta_0}, \\ y &= y_0 + \frac{v_0^2}{2g}(1-\cos\theta_0\sec\theta). \end{aligned} \tag{29}$$

In explicit form, S_1 consists of the catenaries

$$y = y_0 + \frac{v_0^2}{2g}\left[1 - \cosh\frac{2g(x-x_0)}{v_0^2\cos\theta_0} + \sin\theta_0\sinh\frac{2g(x-x_0)}{v_0^2\cos\theta_0}\right]. \tag{30}$$

These are catenaries of the reversed gravitational field of force. If v_0^2 is taken to be negative, then we obtain the catenaries of the given gravitational field of force.

Finally substitute $k=-2$ into (22). The system S_{-2} of brachistochrones is

$$(31)\qquad \begin{aligned} x &= x_0 - \frac{v_0^2}{4g\cos^2\theta_0}(2\theta_0 + \sin 2\theta_0) + \frac{v_0^2}{4g\cos^2\theta_0}(2\theta + \sin 2\theta), \\ y &= y_0 + \frac{v_0^2}{2g}\left[1 - \left(\frac{\cos\theta}{\cos\theta_0}\right)^2\right]. \end{aligned}$$

These are ordinary cycloids.

References

1. E. Kasner, Differential geometric aspects of dynamics, Amer. Math. Soc. Colloquium Publ. 1913, 1934, 1938. See especially pp. 91–97.

2. See the following series of papers by Kasner and De Cicco: Physical curves, Proc. Nat. Acad. Sci., 34, 68 (1948); A generalized theory of dynamical trajectories, Trans. Amer. Math. Soc. 54, 23–38 (1943). Physical systems of curves in space, Proc. Nat. Acad. Sci. 35, 106–108 (1949).

3. Sir Edmund Whittaker, Analytical dynamics, Dover Publications, (1944). See p. 29.

4. J. F. Ritt, Integration in finite terms, Columbia University Press, (1948). See pp. 37–39.

PLANE AREAS BY COMPLEX INTEGRATION

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1. Introduction. The purpose of this paper is to show how the theory of functions of a complex variable can be employed to determine the area enclosed by simple closed curves in the complex plane. The real line integral

$$(1)\qquad A(C) = 1/2\int_C [-y\,dx + x\,dy],$$

which represents the area in the real plane enclosed by the simple closed curve C, is very difficult, if not impossible, to evaluate in practice except for the simpler curves. We shall show how this line integral can be expressed as a complex integral of a very simple type. Under certain conditions the evaluation of this complex integral can be greatly facilitated by the theory of residues and conformal transformations. In this way the problem of determining the area enclosed by a simple closed curve is made to depend upon the theory of residues and conformal mapping, like so many other problems in applied mathematics. This application of complex integration seems to have been overlooked in the literature.

We shall illustrate the facility of this procedure by finding the area of any Joukowski aerofoil section. The result is a remarkable property of these aerofoils.

2. Statement of the problem. Suppose that the region whose area we seek is enclosed by the simple closed curve C, whose equation is

$$\bar{z} = f(z), \tag{2}$$

in the isotropic coordinates $(z, \bar{z})$ where $z=x+iy$ and $\bar{z}=x-iy$. Since

$$x = (z+\bar{z})/2, \qquad y = (z-\bar{z})/2i,$$

any function of the variables (x, y) can be expressed in terms of z and $\bar{z}$. For example, any circle with center at (h, k) and radius r can be represented by the equation

$$z\bar{z} - \overline{P}z - P\bar{z} + K = 0, \quad \text{or} \quad \bar{z} = (\overline{P}z - K)/(z - P), \tag{3}$$

where $P=h+ik$, $\overline{P}=h-ik$, and $K=h^2+k^2-r^2$.

Consider the integral

$$\int_C \bar{z}dz = \int_C (xdx + ydy) + i\int_C (-ydx + xdy),$$

where we assume that the integral is taken along the closed curve C in the positive sense. Since C is closed the first integral in the second member of the last equation is zero, and the second integral is $2i$ times the area integral (1). Therefore, we have

$$A(C) = 1/2i \int_C \bar{z}dz, \tag{4}$$

as a formula for the area enclosed by the curve C. This formula can be expressed in terms of z alone for a given curve C by substituting for $\bar{z}$ from the equation (2) of C, which gives

$$A(C) = 1/2i \int_C f(z)dz. \tag{5}$$

Now, if we assume that the function $f(z)$ is single-valued and meromorphic interior to C and analytic on C, then we may apply the theory of residues to the evaluation of the integral (5).

As a simple example, let us find the area of the circle (3) by this method. In this case, $f(z)=(\overline{P}z-K)/(z-P)$ which is single-valued and meromorphic interior to C, the only singularity being a simple pole at $z=P$, and is analytic on C. Therefore, from the theory of residues, we have

$$\begin{aligned} A(C) &= 1/2i \int_C \frac{(\overline{P}z - K)dz}{z - P} \\ &= \left[2\pi i \times \operatorname*{Res}_{z=P} \frac{\overline{P}z - K}{z - P}\right]/2i \\ &= \pi(\overline{P}P - K) = \pi r^2. \end{aligned}$$

The hypothesis that $f(z)$ in equation (2) is single-valued involves a considerable restriction on the curve C. For example, the equation (2) for the simple ellipse $x^2+4y^2=4$ is

$$\bar{z} = [5z \pm 4(z^2 - 3)^{1/2}]/3$$

and $f(z)$ is not single-valued inside the ellipse. Consequently, the theory of residues is not applicable to the evaluation of the integral (5) for this curve. However, this ellipse is transformed into the circle C' with equation $z'\bar{z}'=9/4$ by means of the transformation

$$z = z' + \frac{3}{4z'}.$$

Since this transformation satisfies the reflection principle

$$\bar{z} = \bar{z}' + \frac{3}{4\bar{z}'},$$

the integral (4) is transformed into the integral

$$A(C) = 1/2i \int_{C'} \left(\bar{z}' + \frac{3}{4\bar{z}'}\right)\left(1 - \frac{3}{4z'^2}\right) dz' = 2\pi,$$

which is easily evaluated by the use of residue theory. The details of this example are omitted here since they follow from a more general problem which is treated in the next section. For example, we shall see that the ellipse under discussion is also transformed into the circle $z'\bar{z}'=1/4$ by the above transformation, and that either circle may be used to evaluate the integral.

In general, then, if the function $f(z)$ in the equation (2) of the curve C is either extremely complicated or multiple-valued, we seek a transformation defined by a single-valued and meromorphic function,

$$z = T(z'),$$

which transforms the curve C into a closed curve C' for which the function $F(z')$ in its equation

$$\bar{z}' = F(z')$$

is single-valued. If this transformation satisfies the reflection principle

$$\bar{z} = T(\bar{z}'),$$

the area integral (4) becomes

$$A(C) = 1/2i \int_C \bar{z}dz = 1/2i \int_{C'} T(\bar{z}')T'(z')dz',$$

where $T'(z')$ denotes the derivative of $T(z')$. If the function $F(z')$ is mero-

morphic interior to C' and analytic on C', the theory of residues is applicable to the evaluation of this integral.

Obviously, one might consider any integral

$$\int_C F(z, \bar{z})dz$$

evaluated along a curve C, closed or not, represented in the form (2). Use is made of this concept in those problems where a transformation is made in order to evaluate the area integral (4).

3. Application to area of Joukowski's aerofoil sections. The transformation

$$w = z + a^2/z, \tag{6}$$

where $z=x+iy$, $w=u+iv$, and a is any positive real constant, maps suitably chosen circles in the z-plane into curves in the w-plane which resemble the cross section of a modern aerofoil. Cylinders having these images as sections are known as Joukowski's aerofoils.

Suppose that the circle in the z-plane is given by equation (3) and let L denote the transformed curve in the w-plane. According to Green,* when the circle C passes through the point $A(z=-a)$ and just encloses the point $B(z=a)$ the form of the curve L is that of an aerofoil section. The shape and dimensions of the aerofoil depend upon the position and radius of the circle C. The equation of the curve L is isotropic coordinates is of the form†

$$a_0(w^2\bar{w}^2) + a_1(w^2\bar{w} + w\bar{w}^2) + a_2(w^2\bar{w} - w\bar{w}^2) + a_3(w^2 + \bar{w}^2) + a_4(w^2 - \bar{w}^2) + a_5(w\bar{w}) + a_6(w + \bar{w}) + a_7(w - \bar{w}) + a_8 = 0,$$

where a_i, $i=0, 1, \cdots, 8$ are known functions of a, h, k, and K. The curve L becomes an aerofoil section if h and k are sufficiently small and K is determined so that C passes through the point A.

Obviously, the formula

$$A(L) = 1/2i \int_L \bar{w}dw \tag{7}$$

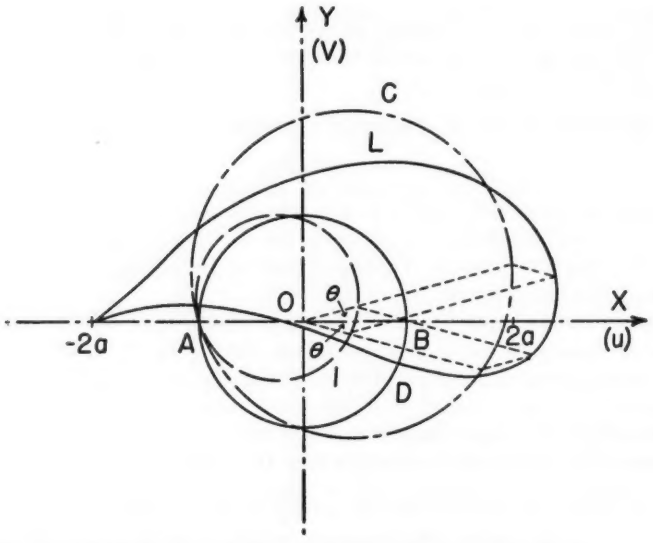
for the area enclosed by L in the w-plane is extremely difficult to evaluate, because the equation of L is complicated and does not define $\bar{w}$ as a single-valued function of w. Therefore, we shall transform the profile L into a circle C in the z-plane by means of the transformation (6).

In order to study this mapping more closely, let D denote the circle with center at the origin and passing through A and B. Let I denote the circle which

* Hydro- and Aerodynamics, Pitman, 1937, pp. 62–67.

† Mancill and Thomas, On the equation of Joukowski's aerofoils, this MONTHLY, vol. 53, 1946, p. 148.

is the inversion of C with respect to D, as shown in the figure. Now, it is easily seen from the geometric construction of the point by point mapping defined by (6) that the region enclosed by L maps into the two regions, not containing the origin, in the z-plane bounded by the circles C and D and by the circle I and D. The profile itself maps into the two circles C and I.



The formula (7) for the area enclosed by the profile L becomes

$$A(L) = 1/2i \int_L \bar{w}dw = 1/2i \int_C \left(\bar{z} + \frac{a^2}{\bar{z}}\right)\left(1 - \frac{a^2}{z^2}\right)dz$$

(8)

$$= 1/2i \int_C \bar{z}dz - a^4/2i \int_C dz/(z^2\bar{z}) + a^2/2i \int_C dz/\bar{z} - a^2/2i \int_C \bar{z}dz/z^2,$$

since the transformation (6) satisfies the reflection principle $\bar{w} = \bar{z} + a^2/\bar{z}$. The first integral on the right in the last expression for $A(L)$ in (8) is the area of the circle C. The second integral is the area of the circle I of inversion of C with respect to D, because

$$a^4/2i \int_{C'} dz/(z^2\bar{z}) = -1/2i \int_{I'} \bar{z}'dz' = A(I)$$

by the transformation $z = a^2/z'$, since the circle C and I are traced in opposite senses by means of this transformation.

Now we wish to show that the other two integrals in (8) are zero. It follows easily that

$$\int_C dz/\bar{z} = \int_C \frac{z - P}{\bar{P}z - K}\,dz = 0,$$

because the point $z = K/\bar{P}$ is outside the circle C and the integrand is a regular function of z in the circle C. The point $z = K/\bar{P}$ is outside the circle C because

$$\begin{aligned} z\bar{z} - \bar{P}z - P\bar{z} + K &= (K/\bar{P})(K/P) - \bar{P}(K/\bar{P}) - P(K/P) + K \\ &= K^2/P\bar{P} - K > 0, \end{aligned}$$

since $K = h^2 + k^2 - r^2 < 0$ under our definition of the aerofoil.

The last integral in (8) is

$$\begin{aligned} a^2/2i \int_C \bar{z}dz/z^2 &= a^2/2i \int_C \frac{\bar{P}z - K}{z^2(z - P)}\,dz \\ &= a^2[2\pi i \times \text{Res. at } z = 0 + 2\pi i \times \text{Res. at } z = P]/2i \\ &= a^2\pi\left(\frac{K - \bar{P}P}{P^2} + \frac{P\bar{P} - K}{P^2}\right) \\ &= 0. \end{aligned}$$

Thus we have proved the following

THEOREM. *The area of any Joukowski aerofoil section is equal to the area of the circle C minus the area of its inversion I, or*

$$\begin{aligned} A(L) &= \pi r^2 - \pi a^4 r^2/(h^2 + k^2 - r^2)^2 \\ &= 4\pi h(h + a)[(h + a)^2 + k^2]/(2h + a)^2. \end{aligned} \tag{9}$$

The last form of the formula for $A(L)$ is obtained by determining r so that the circle C passes through the point $A(z = -a)$.

Two special cases of the aerofoil section are of interest. If $k = 0$, the resulting profile L is symmetric to the u-axis and its area is

$$4\pi h(h + a)^3/(2h + a)^2.$$

If $h = k = 0$, the resulting profile L is a degenerate ellipse, composed of the segment of the u-axis between $z = -2a$ and $z = 2a$, and its area is zero. This checks with formula (9) for this case.

As a final example to verify the formula (9), consider the non-aerofoil section determined by the circle with radius $r \neq a$ and $h = k = 0$. The curve L in the w-plane is the ellipse.

$$\frac{u^2}{(r + a^2/r)^2} + \frac{v^2}{(r - a^2/r)^2} = 1, \tag{10}$$

whose area is

$$\pi(r + a^2/r)(r - a^2/r) = \pi(r^4 - a^4)/r^2.$$

This agrees with the first form of (9) for $h=k=0$.

Also, the example of the ellipse discussed in the preceding section is a special case of these results. For, the ellipse $u^2+4v^2=4$ results from (10) for $a^2=3/4$ and $r=3/2$ or $r=1/2$. In this connection, it should be pointed out that since the profile L maps into both the circles C and I in the z-plane, either circle could have been employed to evaluate the formula (8). The results would have been the same since the roles of C and I would have been merely interchanged.

AN APPLICATION OF DETERMINANTS TO THE PROBABILITY OF MATED PAIRS

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1. Introduction. In this paper we are concerned first with the terms of a determinant, that is, with the combinations of elements which compose them. We wish to know how many terms in the expansion contain a given number of elements from the principal diagonal.

We know that the only term of a first order determinant consists of its one principal diagonal element, while the two terms of a second order determinant consist respectively of a product of elements of and a product of elements exclusive of the principal diagonal. Likewise in a third order determinant it is easy to find that one term contains only principal diagonal elements, three terms contain one, and two terms contain none. For higher order determinants the actual forming and counting of terms is impractical. We consider then that for an nth order determinant all terms containing a specified group of r diagonal elements are the products of those elements by the terms of the $(n-r)$th order minor formed by deleting from the determinant the rows and columns of those elements. If now the minor is made zero-axial, that is, has its diagonal elements replaced by zeros, then the non-vanishing products will contain no diagonal elements except those specified. The problem thus requires us to find the number of non-vanishing terms of a zero-axial determinant.

2. The non-vanishing terms of a zero-axial determinant. Let a determinant of order n in which all but r of the principal diagonal elements are set equal to zero be called a determinant of Type R and let the number of its non-vanishing terms be $T_{n,r}$. We seek a formula for $T_{n,0}$.

Consider first that an nth order determinant of Type I has in its expansion the $T_{n,0}$ non-vanishing terms which contain no diagonal element together with the $T_{n-1,0}$ terms formed when the one non-zero diagonal element is multiplied into the non-vanishing terms of its $(n-1)$st order minor. It follows that

$$T_{n,1} = T_{n,0} + T_{n-1,0}. \tag{1}$$

We digress briefly to remark that in like manner the non-vanishing terms of an nth order determinant of Type R will consist of the $T_{n,0}$ terms containing no elements from the principal diagonal plus the ${}_nC_1T_{n-1,0}$ terms containing precisely one element from the principal diagonal, and so on. Hence,

$$T_{n,r} = T_{n,0} + {}_nC_1T_{n-1,0} + \cdots + {}_nC_rT_{n-r,0}.$$

Let a zero-axial determinant of order $n+1$ be expanded in terms of elements of its first row. In the expansion the upper left-hand element, 0, times its minor contributes no non-vanishing terms. Each of the other n elements of the first row times its nth order minor of Type I contributes $T_{n,1}$ non-vanishing terms. Hence we have the relation

$$T_{n+1,0} = n(T_{n,0} + T_{n-1,0}). \tag{2}$$

From the obvious values $T_{1,0}=0$ and $T_{2,0}=1$ it follows that $T_{3,0}=2$, $T_{4,0}=9$, and $T_{5,0}=44$. These numbers strongly suggest the more easily applicable recursion formula below, which can be established readily by induction.

$$T_{n,0} = nT_{n-1,0} + (-1)^n. \tag{3'}$$

Repeated application of formula (3') gives the relation

$$T_{n,0} = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}\right\}. \tag{4'}$$

The sum in the braces consists of the first n terms in the series expansion for e^{-1}; hence it is alternately larger and smaller than e^{-1}. For $n \neq 0$, $T_{n,0}$ is the integer nearest in value to the product $n!\,e^{-1}$.

The relation (4') may be derived directly from known properties of expansion of determinants by considering the number of terms omitting the first diagonal element, then of those omitting both the first and the second, and so on. A development and discussion of this is given in Metzler's revision of Muir's treatise on determinants [1]. The recursion formula (3') then follows as a corollary from (4').

We have now found an efficient formula for $T_{n-r,0}$ which, as we have seen above, gives the number of terms of an nth order determinant which contain a specified set of r diagonal elements. Since there are ${}_nC_r$ ways of choosing the r elements, it follows that the number of terms containing precisely r diagonal elements is ${}_nC_rT_{n-r,0}$.

If now we use the notation $N_{n,r}$ to indicate the number of terms of an nth order determinant which contain exactly r elements from the principal diagonal, then it is clear that $N_{n,0}=T_{n,0}$ and the formulas (2), (3'), and (4') above can be written in terms of $N_{n,0}$. For the sake of simplification in an expression below let $N_{0,0}=1$. This definition produces the proper value for $N_{1,0}$ when used in the analogue of formula (3'). We summarize results.

THEOREM I. *The number, $N_{n,r}$, of terms of an nth order determinant containing precisely r elements from the principal diagonal is given by the formula*

$$N_{n,r} = {}_nC_r N_{n-r,0}, \tag{5}$$

where $N_{0,0}=1$ and for $n\neq 0$, $N_{n,0}$ is defined by formula

$$N_{n,0} = nN_{n-1,0} + (-1)^n, \tag{3}$$

or by formula

$$N_{n,0} = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\right\}, \tag{4}$$

or by taking the integer nearest in value to the product $n!e^{-1}$.

3. The application to mated pairs. We turn now to the second problem in the paper, the application of Theorem I to a problem in probability. Consider an experiment in which n samples of handwriting are to be matched with photographs of the individuals making them. When correctly matched a pair is said to be mated, otherwise mismated. What is the probability, $p_{n,r}$, due to chance alone, that if n pairs are formed, r will be mates and $n-r$ will be mismates?

The statistical problem is analogous to the following one. A bag contains n balls each marked with a different element of the sequence $A_1, A_2, \cdots, A_n$. Another bag contains n balls each marked with a different element of the sequence $B_1, B_2, \cdots, B_n$. A ball is chosen at random from each bag. The balls in the resulting pair are mates in case they have like subscripts, otherwise mismates. The drawing continues without replacements until all n pairs have been withdrawn from the bags. What is the probability $p_{n,r}$ that in the resulting set of n pairs r pairs are mates and $n-r$ pairs are mismates?

The probability desired is the quotient of the number of sets of n pairs containing exactly r mates by the total number of possible sets of n pairs. The denominator is $(n!)^2$, for there are n^2 ways of drawing balls in the first pair, $(n-1)^2$ ways of drawing balls in the second pair, and so on. Two sets of n pairs are distinct provided the first contains a pair whose subscripts differ from those of any pair in the second set.

To find the numerator we form the square array below in which the principal diagonal elements give mates and the remaining elements give mismates.

$$\begin{vmatrix} A_1B_1 & A_1B_2 \cdots A_1B_n \\ A_2B_1 & A_2B_2 \cdots A_2B_n \\ \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad\quad \cdot \\ A_nB_1 & A_nB_2 \cdots A_nB_n \end{vmatrix}$$

The $n!$ terms of this determinant represent all possible sets of n pairs since each letter with each subscript is used once and only once in each term. How many terms contain exactly r diagonal elements, that is, omit precisely $n-r$ diagonal

elements? The answer is given by $N_{n,r}$ in Theorem I. Each of these sets of n terms can be drawn in $n!$ ways; hence the numerator of the probability fraction is $N_{n,r}n!$ It follows that

$$p_{n,r} = \frac{N_{n,r}n!}{(n!)^2} = \frac{N_{n,r}}{n!} = \frac{N_{n-r,0}}{r!(n-r)!}.$$

A brief computation shows the correctness of the recursion formula (7) stated in the theorem below.

THEOREM II. *The probability $p_{n,r}$ that in drawing n pairs there will be precisely r mates and $n-r$ mismates is given by formula*

$$p_{n,r} = \frac{N_{n-r,0}}{r!(n-r)!}, \tag{6}$$

where $N_{n-r,0}$ is given in Theorem I. Computation may be shortened by noting that $p_{1,1}=1$, $p_{n,0}=N_{n,0}/n!$, and for $n>1$ and $r>0$,

$$p_{n,r} = \frac{p_{n-1,r-1}}{r}. \tag{7}$$

Using the expression in (4) for $N_{n,0}$ we have the formula

$$p_{n,r} = \frac{1}{r!}\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^{n-r}}{(n-r)!}\right\}. \tag{8}$$

The expansion in the braces differs from the value of e^{-1} by an amount less than $1/(n-r+1)!$. Hence the approximation

$$p_{n,r} = \frac{e^{-1}}{r!} \tag{9}$$

differs from the correct value by an error less than

$$E = \frac{1}{r!(n-r+1)!}.$$

It is evident that once the desired number of correct decimal places in expressing the probability has been determined, then the "long method" of calculation by Theorem II or formula (8) may be replaced, after certain values of n, by the "short method" of formula (9). The following table indicates some degrees of accuracy.

Value of n	Number of correct decimal places given by formula (9)
10	4
15	8
20	13
25	18

Inasmuch as such experiments are seldom done with less than ten and usually more than twenty-five balls in each bag, the "short method" will be satisfactory generally for computation. Results are summarized in the following theorem.

THEOREM III. *For values of n greater than* 10 *the probability $p_{n,r}$ that in drawing n pairs there will be precisely r mates and $n-r$ mismates is given correct to at least $n-7$ decimal places by the approximation*

$$p_{n,r} = \frac{e^{-1}}{r!}. \tag{9}$$

If now all members of a class of s students are given n samples of handwriting and asked to match the samples with n photographs shown seriatim on a screen, each photograph appearing once and only once, what is the probability $p_{s,t,r}$ that exactly t of the students will mate r pairs and mismate $n-r$ pairs? The answer to this problem is given by the well-known formula

$$p_{s,t,r} = {}_sC_t(p_{n,r})^t(1 - p_{n,r})^{s-t}, \tag{10}$$

where $p_{n,r}$ is found by use of Theorem II or III.

4. Further extensions to statistical theory. The third part of the paper will tell of other studies of this problem and indicate further statistical extensions. Two discussions of interest to the elementary student are those by Whitworth and Cheney. Both considered the problem of derangements of n ordered objects. W. A. Whitworth [2] called the number $N_{n,0}$ (or $T_{n,0}$) by the name "subfactorial n," defined it by use of formula (3), stated recursion relation (2), and worked out a table of values of $N_{n,0}$ for $n=1, \cdots, 12$. He developed formula (4), noted the approximation of the number value of $N_{n,0}$ by $n!e^{-1}$, and in a problem used relation (5). Whitworth also considered a number of related problems which are of interest but not germane to our discussion. By actual listing and counting of possible derangements, Professor W. F. Cheney, Jr.,* of the University of Connecticut made a table of values of $N_{n,r}$ for n and r between 1 and 10. Using the relations evident in the table he developed the recursion formula (3) and formulas (4), (5), (6), (8), and (9).

Psychologists would find a discussion by D. W. Chapman [3] couched in familiar terms. He considered an x and a y series of t elements each in which the order of the x series remains fixed, but that of the y series may be varied. If precisely s of the elements are to be mated, then s of the y series are also fixed, and the remaining $t-s$ are capable of permutation. From the $(t-s)!$ permutations of the remaining elements must be removed all those in which a y element is mated with its corresponding x element, since the inclusion of these would give arrangements in which $s+1$ of the x and y elements are mated. The permutations to be removed are first reckoned to be $(t-s)!$ in number. That is, any of the $t-s$ elements can be the one to mate, and for each mated

* Mathematics Club Lecture, Connecticut College, 1946.

one there are $(t-s-1)!$ permutations of the remaining elements. But this number of permutations includes also the number of those arrangements with two mated pairs, each counted twice, once for each pair; those containing three mated pairs; *etc.* A series of corrections is thus necessary. When these are made, and when t and s are replaced by n and r, respectively, formula (8) is derived. Chapman then worked out a table correct to four decimal places of the numerical values of the probabilities for $r=0, 1, \cdots, 10$ and $n=1, 2, \cdots, 8$, and $\geqq 9$.

In a later article [4] he generalized the problem to the case in which the x and y series are of different lengths so that one contains extraneous items with no mates in the shorter series. The expression for the probability that a single random arrangement will result in s correct matchings again reduces for our simpler case to the analogue of formula (8).

The remaining discussions are of particular interest to the mathematical statistician. A generalization of our simple problem given by T. N. E. Greville [5] considered the case in which $i_1, i_2, \cdots, i_t$ of the balls in the first bag and $j_1, j_2, \cdots, j_t$ of the balls in the second bag all have subscripts $1, 2, \cdots, t$, respectively, where $i_1+i_2+\cdots+i_t=j_1+j_2+\cdots+j_t=n$. Greville determined the number W_s of sets of n pairs having s mates and $n-s$ mismates. In order to apply his results to our problem take the i's and j's each equal to 1 and the c's each equal to 1 or 0. His formula (3) reduces to

$$V_s = (n-s)!{}_nC_s = \frac{n!}{s}.$$

Replacing W_s and s by our equivalent notation $N_{n,r}$ and r, Greville's formula (4) becomes

$$N_{n,r} = \frac{n!}{r!}\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^{n-r}}{(n-r)!}\right\}.$$

This is our formula (5) when $N_{n-r,0}$ is replaced by the value given in (4).

S. S. Wilks [7] used a different analysis to find the same result. For our simple problem his $n_{dj}=1$ where $d=1, 2$, and $j=1, 2, \cdots, k=n$, so that each $s_i=1$ or 0. His formula (a) reduces to $N=(n!)^2$, the denominator of the probability fraction. For $h=r$, his formulas (h) and (i) combine to give the numerator of the probability fraction $N(r)$, when $N_{n,0}$ is given by formula (4).

$$N(r) = n!{}_nC_r\sum_{g=0}^{n-r}(-1)^{n-r-g}{}_{n-r}C_g g! = n!{}_nC_r N_{n-r,0}.$$

Many other aspects of the problem immediately present themselves. What is the significance of the mean number of correct matchings as the result of i independent trials? What is the nature of the probability function? What are the formulas for moments? What results are obtained when there are three or more decks of arbitrary composition or when all decks but one have fixed

orders in relation to each other? Readers interested in these and other pertinent extensions of the simple problem will profit by further reading of the works of Greville [6] and Wilks as well as an article by I. L. Battin [8] and the papers listed in his careful bibliography. Finally, I. Kaplansky and J. Riordan [9] applied symbolic methods to the problem of multiple matchings which resulted in a simplification of analysis and consequent expedition in obtaining results.

References

1. Thomas Muir and William H. Metzler, *A treatise on the theory of determinants*, New York, 1933, ♯ 124–130.
2. William Allen Whitworth, *Choice and chance*, New York, 1925, pp. 107–110.
3. D. W. Chapman, *American Journal of Psychology*, vol. 46, 1934, pp. 287–298.
4. D. W. Chapman, *Annals of Mathematical Statistics*, vol. 6, 1935, pp. 85–95.
5. T. N. E. Greville, *Annals of Mathematical Statistics*, vol. 12, 1941, pp. 350–354.
6. T. N. E. Greville, *Annals of Mathematical Statistics*, vol. 15, 1944, pp. 432–434.
7. S. S. Wilks, *Mathematical Statistics*, Princeton, 1944, pp. 208–213.
8. I. L. Battin, *Annals of Mathematical Statistics*, vol. 13, 1942, pp. 294–305.
9. I. Kaplansky and J. Riordan, *Annals of Mathematical Statistics*, vol. 16, 1945, pp. 272–277.

NORMAL TRIGRADE AND CYCLIC QUADRILATERAL WITH INTEGRAL SIDES AND DIAGONALS[1]

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1. Complete solution, by normal trigrade, of cyclic quadrilateral with integral sides and diagonals. We will here indicate how one obtains *all* the cyclic quadrilaterals with integral sides and diagonals. We will show, in fact, that this problem is equivalent to that of solving the normal trigrade.

Let us recall a definition.[2] The diophantine system

$$\sum_{i=1}^{4} A_i^k = \sum_{i=1}^{4} B_i^k,$$

where the exponent k takes successively the values 1, 2, 3, is called the normal trigrade, and we represent it by the abridged notation

$$A_1, A_2, A_3, A_4 \overset{3}{=} B_1, B_2, B_3, B_4. \tag{1}$$

Before showing how to obtain all the solutions of (1) we effect several transformations on the unknowns.

[1] Translated from the French by Howard Eves.

[2] For notation and definition of multigrades see the author's Mehrgradige Gleichungen, Noordhoff, Groningen, 1944.

First of all, by a general theorem on multigrades of order n, we are always able to put the trigrade system (1) in a form in which the sum of the terms in each member is zero; one merely subtracts $S_1/4$ from each term, where

$$S_1 = \sum_{i=1}^{4} A_i = \sum_{i=1}^{4} B_i.$$

We obtain

$$\begin{aligned} &3A_1 - A_2 - A_3 - A_4, \quad -A_1 + 3A_2 - A_3 - A_4, \\ &\quad -A_1 - A_2 + 3A_3 - A_4, \quad -A_1 - A_2 - A_3 + 3A_4 \\ &\overset{3}{=} 3B_1 - B_2 - B_3 - B_4, -B_1 + 3B_2 - B_3 - B_4, \\ &\quad -B_1 - B_2 + 3B_3 - B_4, \quad -B_1 - B_2 - B_3 + 3B_4. \end{aligned}$$

An obvious linear change gives the simpler form

$$A_1', A_2', A_3', \quad -(A_1' + A_2' + A_3') \overset{3}{=} B_1', B_2', B_3', \quad -(B_1' + B_2' + B_3').$$

Next we introduce new unknowns by means of the linear system

$$\begin{aligned} a_1 &= A_2' + A_3', \quad & b_1 &= B_2' + B_3', \\ a_2 &= A_3' + A_1', \quad & b_2 &= B_3' + B_1', \\ a_3 &= A_1' + A_2', \quad & b_3 &= B_1' + B_2', \end{aligned}$$

from which one easily eliminates the A_i' and the B_i'. This linear transformation finally reduces the solution of (1) to that of the system

$$\begin{aligned} a_1^2 + a_2^2 + a_3^2 &= b_1^2 + b_2^2 + b_3^2, \\ a_1a_2a_3 &= b_1b_2b_3, \end{aligned} \tag{2}$$

of which the complete solution[3] is given by the following formulas of sixth degree in six arbitrary parameters:

$$\begin{aligned} a_1/r = b_1/s &= p(p^2 - q^2)m^2 - 2r(p^2 - q^2)mn + p(r^2 - s^2)n^2, \\ a_2/ps = b_2/qr &= (p^2 - q^2)m^2 - (r^2 - s^2)n^2, \\ a_3/q = b_3/p &= r(p^2 - q^2)m^2 - 2p(r^2 - s^2)mn + r(r^2 - s^2)n^2. \end{aligned}$$

Another linear transformation

$$\begin{aligned} a_1 + b_1 &= \mu, \quad & a_1 - b_1 &= \nu, \\ b_2 + a_2 &= \pi, \quad & b_2 - a_2 &= \kappa, \\ b_3 &= a_3 = \rho, \quad & b_3 - a_3 &= \sigma, \end{aligned}$$

leads to the diophantine system, equivalent to (2) and consequently to (1),

[3] *Loc. cit.*, pp. 36–37.

$$\mu\nu = \pi\kappa + \rho\sigma,$$
$$\mu(\pi\sigma + \kappa\rho) = \nu(\pi\rho + \kappa\sigma).$$

But these two relations characterize[4] the cyclic quadrilateral having π and κ, ρ and σ as pairs of opposite sides, and μ and ν as diagonals.

One thus obtains all cyclic quadrilaterals whose sides and diagonals can be expressed as integers.

2. Cyclic heronian quadrilaterals. A polygon is said to be *heronian* if its sides and area can be expressed rationally. One is able to obtain, by means of the normal trigrade, a cyclic quadrilateral with integral sides, diagonals, and area.

Here is a solution with four independent parameters m, n, p, q, of system (2). Let a, b, c and α, β, γ represent two pythagorean triples, so that we have

$$a^2 + b^2 = c^2 \quad \text{and} \quad \alpha^2 + \beta^2 = \gamma^2,$$

where

$$a = m^2 - n^2, \qquad b = 2mn, \qquad c = m^2 + n^2,$$
$$\alpha = p^2 - q^2, \qquad \beta = 2pq, \qquad \gamma = p^2 + q^2$$

the second member of the expressions for a, b, c being able to be multiplied by a factor of proportionality l, those for α, β, γ by a factor λ. We then have

$$a_1 = (a + b)(\alpha + \beta), \qquad b_1 = (a - b)(\beta - \alpha),$$
$$a_2 = (a - b)\gamma, \qquad b_2 = (a + b)\gamma,$$
$$a_3 = (\beta - \alpha)c, \qquad b_3 = (\alpha + \beta)c.$$

We may then take

$$\mu = a\beta + b\alpha, \qquad \nu = a\alpha + b\beta,$$
$$\pi = a\gamma, \qquad \kappa = b\gamma,$$
$$\rho = c\beta, \qquad \sigma = c\alpha.$$

These are the diagonals (μ, ν) and opposite sides (π, κ), (ρ, σ) of a cyclic quadrilateral. The diagonals are perpendicular to each other,[5] whence, if S is the area of the quadrilateral[6]

$$S = \mu\nu/2 = (a\beta + b\alpha)(a\alpha + b\beta)/2 = (ab\gamma^2 + \alpha\beta c^2)/2.$$

Also,[7] $R = c\gamma/2$, and the circumdiameter of the quadrilateral is integral.

3. Numerical example. Consider the normal trigrade with sum of terms equal to zero:

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[4] *Each* of the relations characterizes the quadrilateral as cyclic. (H. Eves)

[5] Since $\pi^2+\kappa^2=\rho^2+\sigma^2$. (H. Eves)

[6] S is an integer because b and β are each even. (H. Eves)

[7] $4RS=\mu(\pi\sigma+\rho\kappa)$, whence $2R=(\pi\sigma+\rho\kappa)/\nu=c\gamma$. (H. Eves)

$$-167, -71, 97, 141 \overset{3}{=} -183, 1, 13, 169,$$

$$S_1 = 0, \qquad S_2 = 62{,}220, \qquad S_3 = -1{,}299{,}480.$$

The corresponding solution of system (2) may be taken as

$$(a_1, a_2, a_3) = (119, 13, 35), \qquad (b_1, b_2, b_3) = (7, 91, 85).$$

The pairs of opposite sides of the corresponding cyclic quadrilateral may then be taken as (52, 39), (60, 25), the diagonals as 63 and 56, the area $S = 1764 = 42^2$, and $R = 65/2$.[8]

MATHEMATICAL NOTES

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SYSTEMS OF HYPERBOLOIDS AND OF QUARTIC CURVES CIRCUMSCRIBED ABOUT A TETRAHEDRON*

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1. Notation. Following custom, we shall designate by a, a', b, b', c, c' the lengths of the edges BC, DA, CA, DB, AB, DC, and also the dihedral angles along these edges; by A, B, C, D the areas of the faces BCD, CDA, DAB, ABC; and by d_1, d_2, d_3 the distances between the opposite edges a and a', b and b', c and c' of a tetrahedron $T \equiv ABCD$.

2. Circumscribed hyperboloids and quartic curves. We first establish the following theorem.

THEOREM. *If the planes BCP, CDP, BDP, ADP, ACP, ABP, which pass through the edges of a tetrahedron T and through an arbitrary point P, cut the opposite edges DA, AB, AC, BC, BD, CD in N, U, S, M, Q, W, then the radical axes of the triples of spheres $(ABCN, ACDU, ABDS)$, $(BCDU, BACQ, BADM)$, $(CBDS, CABW, CADM)$, $(DBCM, DACQ, DABW)$ form a hyperbolic group of lines.*

For, if $S = 0$ and $\pi = 0$ are the equations of the sphere $ABCD$ and of the plane at infinity, those of the spheres $DBCN$, $DACQ$, $DABW$ are

$$(A\alpha + D\delta)S - (D\delta A x a'^2)\pi = 0,$$

$$(B\beta + D\delta)S - (D\delta B y b'^2)\pi = 0,$$

$$(C\gamma + D\delta)S - (D\delta C z c'^2)\pi = 0,$$

[8] This same quadrilateral can be arrived at by starting with the two pythagorean triples (3, 4, 5) and (5, 12, 13). (H. Eves)

* Translated from the French by Howard Eves, Oregon State College.

$\alpha, \beta, \gamma, \delta$ denoting the normal coordinates of the point P with respect to the tetrahedron T. Hence, equations of the radical axes of the triples of spheres under consideration are

$$\Delta_a \equiv Byc^2/(B\beta + A\alpha) = Czb^2/(C\gamma + A\alpha) = Dta'^2/(D\delta + A\alpha),$$
$$\Delta_b \equiv Cza^2/(C\gamma + B\beta) = Dtb'^2/(D\delta + B\beta) = Axc^2/(A\alpha + B\beta),$$
$$\Delta_c \equiv Dtc'^2/(D\delta + C\gamma) = Axb^2/(A\alpha + C\gamma) = Bya^2/(B\beta + C\gamma),$$
$$\Delta_d \equiv Axa'^2/(A\alpha + D\delta) = Byb'^2/(B\beta + D\delta) = Cxc'^2/(C\gamma + D\delta).$$

It follows [1] that these four lines form a hyperbolic group.

In order that the lines $\Delta_a, \Delta_b, \Delta_c, \Delta_d$ be concurrent, it is necessary and sufficient that the point P be on the quartic curve

$$(1) \qquad \begin{aligned}(A\alpha + D\delta)(B\beta + C\gamma)/(aa')^2 &= (A\alpha + C\gamma)(B\beta + D\delta)/(bb')^2 \\ &= (A\alpha + B\beta)(C\gamma + D\delta)/(cc')^2,\end{aligned}$$

which passes through the vertices A, B, C, D, and through the harmonic conjugates of the centroid G of T with respect to the medians AG_a, BG_b, CG_c, DG_d of T as segments.

Curve (1) belongs to the pencil of quartic curves

$$(2) \qquad \begin{aligned}(A\alpha + D\delta)(B\beta + C\gamma)/m &= (A\alpha + C\gamma)(B\beta + D\delta)/n \\ &= (A\alpha + B\beta)(C\gamma + D\delta)/p,\end{aligned}$$

where m, n, p are arbitrary parameters. This pencil of curves passes through the eight vertices of the parallelepiped (P) circumscribed about the tetrahedron T; for the planes $A\alpha + D\delta = 0$ and $B\beta + C\gamma = 0$ are parallel, *etc.* Any curve $\mathcal{B}$ of this pencil is, then, the common intersection of three second degree cylinders having for axes the bimedians of T (obtained by setting equal in pairs the three fractions (2)) and of a second degree cone with vertex at G and having for equation

$$(n - p)Q_1 + (p - m)Q_2 + (m - n)Q_3 = 0,$$

where Q_1, Q_2, Q_3 are the numerators of the fractions (2).

3. Another definition of $\mathcal{B}$. Let us denote by x and x' the distances of a point from those faces of (P) which pass through the edges a and a', and similarly define y and y', z and z'. Then we have

$$A\alpha + D\delta = x(A^2 + D^2 - 2AD\cos a)^{1/2} = (x/2)aa' \sin(a, a') = 3Vx/d_1,$$
$$B\beta + C\gamma = x'(B^2 + C^2 - 2BC\cos a')^{1/2} = (x'/2)aa' \sin(a, a') = 3Vx/d_1.$$

The numerators of the fractions (2) are thus proportional to

$$xx'/d_1^2, \qquad yy'/d_2^2, \qquad zz'/d_3^2.$$

Consequently, *the quartic curve $\mathcal{B}$ is the locus of points for which the products of the distances to the pairs of parallel faces of the parallelepiped* (P) *divided by the squares of the distances between these pairs of faces are proportional to* m, n, p.

These distances can, moreover, be taken either perpendicular to the faces or in an arbitrary direction, for x, x', d_1 vary proportionally when one changes the direction. For example, the distances may follow the directions of those edges of (P) not parallel to the pairs of respective faces.

4. Special cases. When m, n, p are proportional to $(aa')^2$, $(bb')^2$, $(cc')^2$, one obtains the quartic curve (1). In order to characterize it in the pencil of curves $\mathcal{B}$, it suffices to fix one point.

If the tetrahedron T is *orthocentric*, a quartic curve $\mathcal{B}$ passes through the orthocenter H, and consequently through the circumcenter, because, for H, we have

$$xx' = yy' = zz'.$$

By performing two dilatations of the parallelepiped (P) relative to any tetrahedron T in the directions of the axes of (P), *i.e.*, the bimedians of T, in such a fashion as to render these bimedians equal to each other, one obtains an orthocentric tetrahedron T'. Under the transformation, a quartic $\mathcal{B}$ of T goes into a quartic $\mathcal{B}'$ of T'. The correspondent of the orthocenter of T' gives, then, a point through which $\mathcal{B}$ passes in the general tetrahedron.

If the tetrahedron T is *isodynamic*, the denominators of the fractions (1) are equal, and *the quartic curve* (1) *degenerates into the four diagonals of* (P), (*medians of* T), *and passes through the centroid* G.

5. Note. The fact that in the quartic curve (1), aa', bb', cc' are proportional to the sides of antiparallel sections of T [2] ought perhaps to confer some special properties with respect to these sections, and consequently to the *second Lemoine point* of T.

References

1. Koehler, Exercises de Géométrie, t. II (coordonées tétraèdriques).
2. Rouché et de Comberousse, Traité de Géométrie, deuxième partie (1891), p. 599.

CHANGE IN POTENTIAL DUE TO A DIELECTRIC SPHERE

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1. Introduction. When a sphere of radius a, filled with homogeneous dielectric of specific inductive capacity k, is placed with its center at the origin of coordinates in any electrostatic field whose potential function is $\phi(x, y, z)$, having no singularities inside or on the sphere, then, by considering the boundary values at the surface of the sphere, we can determine the potential functions inside and outside the sphere in terms of $\phi(x, y, z)$. These potential functions are respectively

$$(1) \qquad \phi_1 \equiv \frac{2}{(k+1)}\phi(x, y, z) + \frac{(k-1)}{(k+1)^2}\int_0^1 t^{-k/(k+1)}\phi(xt, yt, zt)dt,$$

and

$$\phi_2 \equiv \phi(x, y, z) - \frac{(k-1)}{(k+1)} \frac{a}{r} \phi(x_1, y_1, z_1) + \frac{(k-1)}{(k+1)^2} \frac{a}{r} \int_0^1 t^{-k/(k+1)} \phi(x_1 t, y_1 t, z_1 t) dt, \tag{2}$$

where

$$x_1 = \frac{a^2 x}{r^2}, \qquad y_1 = \frac{a^2 y}{r^2}, \qquad z_1 = \frac{a^2 z}{r^2}, \qquad r^2 = x^2 + y^2 + z^2.$$

It is assumed that there are no other boundaries present.

2. Proof. To prove (1) and (2), we expand $\phi(x, y, z)$ in the form

$$\phi \equiv \sum_{n=0}^{\infty} A_n r^n S_n,$$

where S_n is, as usual, a surface harmonic.

When the sphere is placed with its center at the origin, the potential function ϕ_1 inside the dielectric is given by

$$\phi_1 \equiv \sum_{n=0}^{\infty} B_n r^n S_n, \qquad r \leqq a.$$

Outside the dielectric the potential function ϕ_2 will be given by

$$\phi_2 \equiv \phi + \sum_{n=0}^{\infty} C_n r^{-n-1} S_n, \qquad r \geqq a,$$

the latter term representing the interference potential of the dielectric sphere (it must be of this form because it satisfies Laplace's equation and also tends to zero as $r \to \infty$).

The conditions to be satisfied are

$$\phi_1 = \phi_2 \qquad \text{on } r = a,$$

and

$$k \frac{\partial \phi_1}{\partial r} = \frac{\partial \phi_2}{\partial r} \qquad \text{on } r = a.$$

Thus

$$\sum_{n=0}^{\infty} B_n a^n S_n = \sum_{n=0}^{\infty} A_n a^n S_n + \sum_{n=0}^{\infty} \frac{C_n S_n}{a^{n+1}},$$

and

$$k\sum_{n=1}^{\infty} nB_n a^{n-1}S_n = \sum_{n=1}^{\infty} nA_n a^{n-1}S_n - \sum_{n=1}^{\infty} \frac{(n+1)C_n S_n}{a^{n+2}}.$$

Therefore

$$B_n a^{2n+1} - C_n = A_n a^{2n+1},$$

and

$$knB_n a^{2n+1} + (n+1)C_n = nA_n a^{2n+1}.$$

From these two equations we obtain

$$B_n = \frac{(2n+1)A_n}{n(k+1)+1} \quad \text{and} \quad C_n = \frac{-n(k-1)A_n a^{2n+1}}{n(k+1)+1}.$$

Writing

$$B_n = \frac{2A_n}{(k+1)} + \frac{(k-1)}{(k+1)} \frac{A_n}{(nk+n+1)},$$

we see that

$$\phi_1 = \frac{2}{(k+1)} \sum_{n=0}^{\infty} A_n r^n S_n + \frac{(k-1)}{(k+1)^2} \sum_{n=0}^{\infty} \frac{A_n r^n S_n}{n+1-\frac{k}{k+1}}.$$

Now $\phi(x, y, z) \equiv \sum_{n=0}^{\infty} A_n r^n S_n$ is a sum of terms homogeneous in x, y, z of degree n, $n=0, 1, 2, \cdots$, so that (1) holds.

Again writing C_n in the form

$$C_n = -\frac{(k-1)}{(k+1)} A_n a^{2n+1} + \frac{(k-1)}{(k+1)} \frac{A_n a^{2n+1}}{(nk+n+1)},$$

$$= -\frac{(k-1)}{(k+1)} A_n a^{2n+1} + \frac{(k-1)}{(k+1)^2} \frac{A_n a^{2n+1}}{\left(n+1-\frac{k}{k+1}\right)},$$

we see that

$$\sum_{n=0}^{\infty} C_n \frac{S_n}{r^{n+1}} = -\frac{(k-1)}{(k+1)} \sum_{n=0}^{\infty} \frac{A_n a^{2n+1}}{r^{n+1}} + \frac{(k-1)}{(k+1)^2} \sum_{n=0}^{\infty} \frac{A_n a^{2n+1}}{r^{n+1}\left(n+1-\frac{k}{k+1}\right)}.$$

If (x_1, y_1, z_1) is the inverse point of (x, y, z) with regard to the sphere, then $rr_1 = a^2$, and so

$$\sum_{n=0}^{\infty} \frac{C_n S_n}{r^{n+1}} = \sum_{n=0}^{\infty} \frac{C_n S_n r_1^{n+1}}{a^{2n+2}}$$

$$= -\frac{(k-1)}{(k+1)} \sum_{n=0}^{\infty} \frac{A_n r_1^{n+1} S_n}{a} + \frac{(k-1)}{(k+1)^2} \sum_{n=0}^{\infty} \frac{A_n r_1^{n+1} S_n}{a\left(n+1-\frac{k}{k+1}\right)}$$

$$= -\frac{(k-1)}{(k+1)} \frac{a}{r} \sum_{n=0}^{\infty} A_n r_1^n S_n + \frac{(k-1)}{(k+1)^2} \frac{a}{r} \sum_{n=0}^{\infty} \frac{A_n r_1^n S_n}{\left(n+1-\frac{k}{k+1}\right)}$$

$$= -\frac{(k-1)}{(k+1)} \frac{a}{r} \phi(x_1, y_1, z_1) + \frac{(k-1)}{(k+1)^2} \frac{a}{r} \int_0^1 t^{-k/(k+1)} \phi(x_1 t, y_1 t, z_1 t)\, dt.$$

Thus (2) holds.

Equations (1) and (2) give the new potentials inside and outside the sphere respectively.

For a solid spherical conductor, we let $k \to \infty$ so that

$$\phi_1 = 0, \qquad \phi_2 = \phi(x, y, z) - \frac{a}{r} \phi(x_1, y_1, z_1).$$

3. Examples. We present two examples.

(a) Let a charge e be placed at $A(c, 0, 0)$, $c > a$. Here

$$\phi(x, y, z) \equiv \frac{e}{[(x-c)^2 + y_1^2 + z_1^2]^{1/2}}.$$

The second term in ϕ_2 is

$$-\frac{(k-1)}{(k+1)} \cdot \frac{ea}{r} \cdot \frac{1}{[(x_1 - c)^2 + y^2 + z^2]^{1/2}}$$

$$= -\frac{(k-1)}{(k+1)} \cdot \frac{ea}{c} \cdot \frac{1}{\left[\left(x - \frac{a^2}{c}\right)^2 + y^2 + z^2\right]^{1/2}}.$$

The last term in ϕ_2 is

$$\frac{(k-1)}{(k+1)^2} \frac{a}{r} \int_0^1 \frac{t^{-k/(k+1)}\, dt}{[(x_1 t - c)^2 + y_1^2 t^2 + z_1^2 t^2]^{1/2}}$$

$$= \frac{(k-1)}{(k+1)^2} \frac{ea}{c} \int_0^1 \frac{t^{-k/(k+1)}\, dt}{\left[\left(x - \frac{a^2 t}{c}\right)^2 + y^2 + z^2\right]^{1/2}}.$$

Thus the image system of a charge e with regard to a dielectric sphere is a charge of strength $-(k-1)ea/(k+1)c$ at A_1, the inverse point of A, and a line distribution of sources extending from the center 0 to A_1, the strength per unit length of which, at a distance b from 0, is

$$\frac{(k-1)}{(k+1)^2}\frac{e}{a}\left(\frac{cb}{a^2}\right)^{-k/(k+1)}.$$

(b) As a hydrodynamical problem, consider a sphere of radius a placed with its center at the origin in a fluid whose velocity potential is $\phi(x, y, z)$. These conditions correspond to $k=0$ in the previous work for the velocity potential outside the sphere. From (2) we see that the new velocity potential is

$$\phi(x, y, z) + \frac{a}{r}\phi(x_1, y_1, z_1) - \frac{a}{r}\int_0^1 \phi(x_1 t, y_1 t, z_1 t)dt.*$$

CLASSROOM NOTES

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THE DIOPHANTINE EQUATION OF A CARELESS ERROR

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1. Introduction. A rather common error committed by students on all levels from the elementary school to sophomore mathematics and beyond takes the general form: $(a+b)/(c+d)=a/c+b/d$. During the later stages, this mistake is somewhat unusual when the quantities involved are numerical, but is still frequently applied to algebraic or analytical expressions which later in the problem may be numerically evaluated. Many instructors have had the somewhat disheartening experience of having students achieve the correct result after commission of this gross error. This will be the case if $cd\neq 0$, and $ad^2+bc^2=0$ either identically or for the applicable numerical values.

If a, b, c, d are integral, the Diophantine equation, $ad^2=-bc^2$, results. This equation is of the general form, $\prod x_i^{\alpha_i}=\prod y_i^{\beta_i}$, to which the general method of E. T. Bell† is immediately applicable. Although this method has been termed

* P. Weiss, *On hydrodynamical images. Arbitrary irrotational flow disturbed by a sphere*, Cambridge Philos. Soc. vol. 40, 1945, pp. 259–261.

† Bell, E. T., Diophantine equations from algebraic invariants and covariants. Annals of Math. vol. 34 (2nd ser.) 1933, pp. 450–60.

the most general one for the solution of nonlinear Diophantine equations, the instances of interesting or motivated equations to which it is directly applicable are quite rare. Normally, an equation must be first drastically rearranged and substitutions made to bring it into the desired form. It is therefore believed that instructors who wish to include some introduction to modern methods in Diophantine Equations in courses in Elementary Number Theory will be interested in this example.

2. Solution of $ad^2=-bc^2$ in integers. The general theorem states: All solutions of $xy=zw$ are given by $x=ab$, $y=fg$, $z=af$, $w=bg$, with $(b, f)=1$. In the successive applications of this theorem, parameters appearing in the final result will be in Latin letters, temporary parameters in Greek. Following each major step, all g.c.d. conditions applicable will be listed in the parameters valid at that point. A symmetric solution may be obtained as follows:

a) $d^2=\alpha\beta$, $a=\gamma u$, $c^2=\alpha\gamma$, $-b=\beta u$; $(\beta, \gamma)=1$.

b) If $d^2=\alpha\beta$; $d=\delta s=\zeta\theta$, $\alpha=\delta\zeta$, $\beta=s\theta$; $(s, \zeta)=1$ .Then in $\delta s=\zeta\theta$, $s\mid\theta$ and $\zeta\mid\delta$; therefore $\theta=s\kappa$ and $\delta=\zeta\kappa$. We now have: $d=s\zeta\kappa$, $\alpha=\zeta^2\kappa$, $\beta=s^2\kappa$; $(s, \gamma)=(s, \zeta)=(\kappa, \gamma)=1$.

c) Similarly, from $c^2=\alpha\gamma$, $c=t\lambda\mu$, $\alpha=\lambda^2\mu$, $\gamma=t^2\mu$; $(s, t)=(s, \mu)=(\kappa, t)=(\kappa, \mu)=(s, \zeta)=(t, \lambda)=1$.

d) The values of α give: $\lambda^2\mu=\zeta^2\kappa$. Since $(\kappa, \mu)=1$, $\mu=\pm n^2$, $\kappa=\pm r^2$. Thus $\lambda n=\zeta r$ (the case $\lambda n=-\zeta r$ will yield nothing new in the final result), and $(n, r)=1$. Therefore $\zeta=mn$ and $\lambda=mr$, so that:

$$a=n^2t^2u, \qquad b=-r^2s^2u$$

$$c=mn^2rt \qquad d=mnr^2s;$$

$(s, t)=(s, n)=(r, t)=(r, n)=(s, m)=(t, m)=1$. The g.c.d. conditions in the final result serve only to avoid repetition.

If $d^2=\alpha\beta$ and $c^2=\alpha\gamma$ are solved successively, rather than simultaneously as above, a non-symmetric set of solutions is obtained involving one less parameter: $a=u^2v$, $b=-r^2t^2v$, $c=stu$, $d=rst^2$; $(r, s)=(r, u)=(t, u)=1$.

3. Remarks. By virtue of the fact that the basic theorem is valid in any integral domain with unique factorization, the same set of solutions, with appropriate provision for units, is complete in any such domain, e. g., polynomials with coefficients in a field.

The value of the expressions $(a+b)/(c+d)$ and $a/c+b/d$ is, using the symmetric set, $u(nt-rs)/mnr$. It is interesting to note that any rational number, x/y, can be obtained, and, if the trivial restriction, $(x, y)=1$ is made, with the 'common multipliers' u and m each equal to one. It is merely necessary to factor y into co-prime factors n and r, and then solve the linear Diophantine equation $nt-rs=x$ for s and t. This can be done within the g.c.d. conditions since $(x, nr)=1$.

Finally, it may be that the values derived in this note may be of use in con-

vincing those students given to 'proving' theorems by citing special cases, that in mathematics frequently even an infinity of swallows does not make a summer.

NORMAL AND TANGENTIAL ACCELERATION

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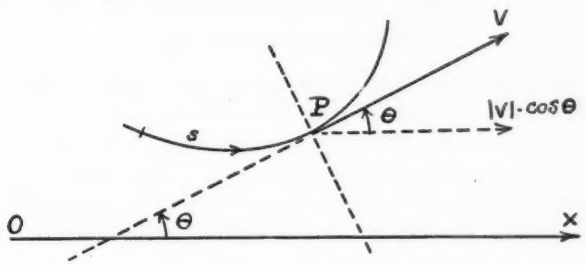
As a first approach to acceleration in the study of plane motion both student and teacher would be happy to avoid the customary routine of converting from components in an arbitrary rectangular reference system to the more useful normal and tangential ones. The following discussion, essentially intrinsic in character, has this appeal.



At a point P of the path let θ be the angle made by the velocity vector V with any direction OX. This angle, measured *from* the direction line OX *to* the velocity vector, is counterclockwise-positive. The component of V in this direction is

$$V_\theta = |V| \cdot \cos\theta.$$

The acceleration component in this same direction is the time derivative:

$$a_\theta = \frac{dV_\theta}{dt} = \frac{d|V|}{dt} \cdot \cos\theta - |V| \cdot (\sin\theta) \cdot \frac{d\theta}{dt}$$

$$= \frac{d|V|}{dt} \cdot \cos\theta - |V| \cdot (\sin\theta) \cdot \frac{d\theta}{ds} \cdot \frac{ds}{dt},$$

or

$$a_\theta = \frac{d|V|}{dt} \cos\theta - |V|^2 \sin\theta \frac{d\theta}{ds}.$$

If we select the direction OX parallel to the tangent by taking $\theta = 0$, we obtain the *tangential acceleration*:

$$a_T = \frac{d|V|}{dt}.$$

Supposing that $d\theta/ds \neq 0$ at P, we now select OX along the normal at P pointing toward the concave side of the curve. If $d\theta/ds > 0$ at P, we have: $\theta = -\pi/2$ and $d\theta/ds = 1/R$ where R is the radius of curvature. If $d\theta/ds < 0$ at P, we have $\theta = \pi/2$ and $d\theta/ds = -1/R$. Then in either case:

$$a_N = \frac{|V|^2}{R},$$

the *normal* component of the acceleration measured in the direction of the center of curvature.

NOTE ON VECTOR PRODUCTS

M. E. SHANKS, Purdue University

Vector analysis books define the vector product geometrically. The definition is justified solely by its later usefulness. Frequently students are repelled by the arbitrariness of the definition and feel that there should be some inherent necessity for such a peculiar product. It is the purpose of this note to supply an algebraic argument, based on a simple physical assumption, which shows that the ordinary product is, to within a multiplicative constant, the only one possible.*

For the following discussion familiarity with the scalar or dot "product" is necessary as well as the fact that the scalar product is invariant under rotation of axes. Guided by a feeling for algebra we seek a definition for a true product, $\mathbf{u}\times\mathbf{v}$, of two vectors $\mathbf{u}$ and $\mathbf{v}$. We use the common notation and denote a right-hand set of orthonormal base vectors by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. We require of our product only the distributive law: $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$, and $(\alpha\mathbf{u})\times(\beta\mathbf{v}) = \alpha\beta\mathbf{u}\times\mathbf{v}$, where α, β are real numbers. Since distributivity is assumed it suffices to define products of base vectors only. Stated otherwise, we seek to make an algebra out of a real 3-dimensional vector space with an inner product, or a real 3-dimensional unitary space.

A vector space becomes an algebra by giving an arbitrary multiplication table for the base vectors. Some restriction must therefore be placed on the product and since vectors are intended for use in physics we examine the physical requirements. Now in physics there are no preferred orientations; therefore, since any right-hand set of base vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ can be transformed by rotation into any other set $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$, it follows that the formula for the product must be the same for all sets of base vectors. It is an easy consequence that, if R is a rotation, then $R(\mathbf{u}\times\mathbf{v}) = (R\mathbf{u}\times R\mathbf{v})$, or *products are invariant under rotation.*

* While this result does not seem to be explicitly stated in the literature on vector analysis, it is implicit in the fact that the basic invariants of the orthogonal group are the scalar product and triple product. See H. Weyl, "The Classical Groups," p. 53.

THEOREM 1. *The only vector product which is invariant under rotation is given by*

$$\mathbf{i}\times\mathbf{j}=\lambda\mathbf{k}=-\mathbf{j}\times\mathbf{i},\qquad \mathbf{j}\times\mathbf{k}=\lambda\mathbf{i}=-\mathbf{k}\times\mathbf{j},$$
$$\mathbf{k}\times\mathbf{i}=\lambda\mathbf{j}=-\mathbf{i}\times\mathbf{k},\qquad \mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0,$$

where λ *is an arbitrary real number.*

Consider $\mathbf{i}\cdot\mathbf{i}\times\mathbf{i}$ and the rotation R: $R\mathbf{i}=-\mathbf{i}$, $R\mathbf{j}=-\mathbf{j}$, $R\mathbf{k}=\mathbf{k}$. Since both scalar and vector products are invariant under rotation, $\mathbf{i}\cdot\mathbf{i}\times\mathbf{i}=R\mathbf{i}\cdot R(\mathbf{i}\times\mathbf{i})=R\mathbf{i}\cdot R\mathbf{i}\times R\mathbf{i}=-\mathbf{i}\cdot\mathbf{i}\times\mathbf{i}$. Hence $\mathbf{i}\cdot\mathbf{i}\times\mathbf{i}=0$. Similarly it follows that $\mathbf{j}\cdot\mathbf{i}\times\mathbf{i}=0$. With a different rotation one obtains $\mathbf{k}\cdot\mathbf{i}\times\mathbf{i}=0$. Therefore $\mathbf{i}\times\mathbf{i}=\mathbf{0}$, and because of the invariance of products under rotation $\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}$.

Consider now $\mathbf{i}\times\mathbf{j}$. We compute $\mathbf{i}\cdot\mathbf{i}\times\mathbf{j}$ using the rotation R: $R\mathbf{i}=-\mathbf{i}$, $R\mathbf{j}=-\mathbf{j}$, $R\mathbf{k}=\mathbf{k}$. Then $\mathbf{i}\cdot\mathbf{i}\times\mathbf{j}=R\mathbf{i}\cdot R(\mathbf{i}\times\mathbf{j})=R\mathbf{i}\cdot R\mathbf{i}\times R\mathbf{j}=-\mathbf{i}\cdot\mathbf{i}\times\mathbf{j}$ and $\mathbf{i}\cdot\mathbf{i}\times\mathbf{j}=0$. Similarly $\mathbf{j}\cdot\mathbf{i}\times\mathbf{j}=0$. Therefore $\mathbf{i}\times\mathbf{j}=\lambda\mathbf{k}$ where λ is a real number.

The rotation R: $R\mathbf{i}=\mathbf{j}$, $R\mathbf{j}=\mathbf{k}$, $R\mathbf{k}=\mathbf{i}$, gives $\mathbf{j}\times\mathbf{k}=\lambda\mathbf{i}$ and $\mathbf{k}\times\mathbf{i}=\lambda\mathbf{j}$.

By the same argument it follows that $\mathbf{j}\times\mathbf{i}=\mu\mathbf{k}$, $\mathbf{k}\times\mathbf{j}=\mu\mathbf{i}$, $\mathbf{i}\times\mathbf{k}=\mu\mathbf{j}$, where μ is real.

Finally, consider $\mathbf{k}\cdot(\mathbf{i}\times\mathbf{j}+\mathbf{j}\times\mathbf{i})$ and the rotation R: $R\mathbf{i}=\mathbf{j}$, $R\mathbf{j}=-\mathbf{i}$, $R\mathbf{k}=\mathbf{k}$. We have $\mathbf{k}\cdot(\mathbf{i}\times\mathbf{j}+\mathbf{j}\times\mathbf{i})=\mathbf{k}\cdot(-\mathbf{j}\times\mathbf{i}-\mathbf{i}\times\mathbf{j})$, whence $\mathbf{k}\cdot(\mathbf{i}\times\mathbf{j}+\mathbf{j}\times\mathbf{i})=0=\lambda+\mu$ or $\mu=-\lambda$. This completes the proof. It may be noted that the only rotations used in the proof are those generated by rotations through 90° about the coördinate axes.

Thus in a real 3-dimensional unitary space products invariant under rotation are essentially unique. It is natural then to inquire into the possibilities when the dimension is not 3. A 1-dimensional space is isomorphic to the real line, and the only "rotation" is the identity. If $\mathbf{i}$ is a unit base vector then $\mathbf{i}\times\mathbf{i}=\lambda\mathbf{i}$ gives a product. But for other dimensions it is easy to prove, using the same arguments as above, the following theorem.†

THEOREM 2. *In a real n-dimensional unitary space, the only product invariant under orthogonal transformations of determinant* 1 *is the trivial one* $u\times v=0$, *if* $n\neq 1, 3$.

ANGLE WITH RESPECT TO A FAMILY OF PLANE CURVES

L. V. TORALBALLA, Marquette University

On writing the equation of the circle $x^2+y^2=a^2$ in the parametric form $x=a\cos\theta$, $y=a\sin\theta$, one observes that the area swept by the radius vector from the parameter value $\theta=0$ to the parameter value $\theta=\theta_0$ is $\frac{1}{2}a^2\theta_0$. One could single out this particular property of θ and consider it the defining property of a generalized angle.

To make the analogy with the above circle more complete we shall consider only one-parameter families of curves. That is to say, we shall deal only with

† In n dimensions a "product" of $n-1$ vectors may be formed which is invariant under the proper orthogonal group. See Weyl, *loc. cit.*, p. 49.

curves having a cartesian equation of the form $h(x, y, a)=0$, where a is the curve family parameter. For the same reason we further confine ourselves to those curves for which $h(x, y, a)$ is homogeneous in x, y, and a. Such curves can always be put in the form $x=a\cdot f(\theta)$, $y=a\cdot g(\theta)$, in which f and g do not involve a.

If f and g are so chosen that $g(0)=0$ and the area swept by the radius vector from $\theta=0$ to $\theta=\theta_0$ is $\frac{1}{2}a^2\theta_0$, then we shall call θ an angle with respect to the curve family $h(x, y, a)=0$. By analogy with the circle $x^2+y^2=a^2$, we may call $f(\theta)$ the "cosine of θ" and $g(\theta)$, the "sine of θ," with respect to the given curve family. From these two we may define the other generalized "trigonometric" functions of θ, in the usual manner. On this basis we may study the "trigonometric" properties of the given curve family. In the special case of the hyperbola $x^2-y^2=a^2$, $f(\theta)$ and $g(\theta)$ may be chosen to be $(e^\theta-e^{-\theta})/2$ and $(e^\theta+e^{-\theta})/2$ respectively. These are respectively called the hyperbolic sine and cosine. In the case of the parabola, the functions $f(\theta)$ and $g(\theta)$ could be called the parabolic sine and cosine respectively.

Let $\rho=F(\phi)$ be the polar equation of the given curve family. For $\theta=0$, $\phi=0$. For $\theta=\theta_0$, let $\phi=\phi_0$. Then the area swept by the radius vector from $\phi=0$ to $\phi=\phi_0$ is

$$A=\frac{1}{2}\int_0^{\phi_0}\rho^2 d\phi$$

$$=\frac{1}{2}a^2\int_0^{\theta_0}[f^2(\theta)+g^2(\theta)]\cdot\left[\frac{f(\theta)g'(\theta)-g(\theta)f'(\theta)}{f^2(\theta)+g^2(\theta)}\right]d\theta\,.$$

Hence, a necessary and sufficient condition that for all values of θ_0, this area will be equal to $\frac{1}{2}a^2\theta_0$ is

$$f(\theta)g'(\theta)-g(\theta)f'(\theta)=1.$$

From the equation $h[af(\theta), ag(\theta), a]=0$, one obtains, on account of the homogeneity, a relation between $f(\theta)$ and $g(\theta)$ which does not involve a. This relation taken together with the above differential relation, leads to an ordinary differential equation involving either only $f(\theta)$ and its derivative $f'(\theta)$ or only $g(\theta)$ and $g'(\theta)$. From either of these, a parametric representation is obtained having the desired properties. The representation is not unique as is shown by the free choice of the constant of integration.

In particular, for the hyperbola $x^2-y^2=a^2$, the corresponding differential equation is

$$f'(\theta)=\sqrt{f^2(\theta)-1}.$$

This yields the parametric equations

$$x=a\cosh\theta;\qquad y=a\sinh\theta.$$

For the ellipse $x^2+2y^2=a^2$, the corresponding representation is

$$x = a\cos(\sqrt{2}\,\theta); \qquad y = \frac{a}{\sqrt{2}}\sin(\sqrt{2}\,\theta).$$

For the hyperbola $x^2-2y^2=a^2$, the representation is

$$x = a\left[\frac{e^{\sqrt{2}\,\theta}-e^{-\sqrt{2}\,\theta}}{2}\right]; \qquad y = a\left[\frac{e^{\sqrt{2}\,\theta}+e^{-\sqrt{2}\,\theta}}{2}\right].$$

For the parabola $y^2=2ax$, the representation is

$$x = \tfrac{1}{2}a\sqrt[3]{36\theta^2}; \qquad y = a\sqrt[3]{-6\theta}$$

For the circle $x=a\cos\theta$, $y=a\sin\theta$, the parameter θ has also the property that the arc length from $\theta=0$ to $\theta=\theta_0$ is $a\theta_0$. One could adopt this property as the defining property of the generalized angle, instead of the property discussed above.

Then for the curve family $h(x, y, a)=0$, of the type described above, one should, obviously, use the differential relation $f'^2(\theta)+g'^2(\theta)=1$, instead of the relation $f(\theta)g'(\theta)-g(\theta)f'(\theta)=1$. However, in general, the functions $f(\theta)$ and $g(\theta)$ thus obtained are more complex than those obtained before.

For an arbitrary curve, it is not possible to choose $f(\theta)$ and $g(\theta)$ so that the area be $\frac{1}{2}a^2\theta_0$ and at the same time the arc length be $a\theta_0$. In fact this can be done only for the family of circles with center at the origin and for the families of parallel straight lines.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 952 [1951, 108]. *Correction*

Change the direction of the inequality signs.

E 961. *Proposed by Leo Moser, Texas Technological College*

Find the probability that if the digits 0, 1, 2, $\cdots$, 9 be placed in random order in the blank spaces of

5_383_8_2_936_5_8_203_9_3_76

the resulting number will be divisible by 396. (By permission of Prof. E. P. B. Umbugio, April 1, 1951.)

E 962. *Proposed by B. H. Brown, Dartmouth College*

A cask of unit volume is full of wine. A man withdraws from the cask an amount a $(0<a<1)$, and then adds amount a of *water*, which is assumed to mix perfectly with the wine. He continues this process b times. He then withdraws amount a and adds amount a of *wine*, continuing this process b times.

(1) Derive a general formula for the amount of wine in the cask after the $2b$ operations.

(2) If $1/a=A$ is an integer, and if B is that corresponding (integral) value of b for which the eventual wine-content is a minimum, so determine A and B that the wine-content is (a) the minimum minimorum (b) the maximum minimorum.

(3) If $A=51$, find B and the wine-content.

(4) Find $\lim_{A\to\infty}(B/A)$.

E 963. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

It had started snowing before noon and three snow plows set out at noon, 1 o'clock, and 2 o'clock, respectively, along the same path. If at some later time they all came together simultaneously, find the time of meeting and also the time it started snowing.

E 964. *Proposed by C. W. Trigg, Los Angeles City College*

(1) Determine the relationship between the sides of a triangle ABC if the line joining the Gergonne point P to the Nagel point Q is parallel to side c.

(2) Show that, given a and b, such a triangle can always be constructed with straight edge and compasses.

(3) Find the smallest scalene triangle of this type with integer sides.

E 965. *Proposed by R. K. Meany, Purdue University*

A function $f(x)$ is continuous in the open interval (a, b). Its derivative, $f'(x)$ exists at every point of (a, b) except perhaps at c, but $\lim_{x\to c} f'(x)$ exists and equals A. Show that then $f'(c)$ also exists and equals A.

SOLUTIONS

Enumeration of Triangles in a Polygon

E 927 [1950, 483]. *Proposed by J. H. Braun, Illinois Institute of Technology*

In a regular polygon of $2n+1$ sides all the diagonals are drawn. Find, as a function of n, the total number of triangles of all shapes and sizes thus formed.

Solution by G. W. Walker, Buffalo, N. Y. If the sides of any triangle are extended, they will intersect the circle circumscribing the polygon in 3, 4, 5, or 6 distinct points, each point a vertex of the polygon, according as the triangle has 3, 2, 1, or 0 corners on the perimeter of the polygon. In the first case, the points may be chosen in C_3^{2n+1} ways. In the second case, the 4 points may be chosen in C_4^{2n+1} ways, and the 2 (of the 4) may be chosen (cyclically adjacent)

for the vertices of the triangle in 4 ways. In the third case, the 5 points may be chosen in C_5^{2n+1} ways, and the point for the vertex of the triangle may be chosen (from the 5) in 5 ways. In the last case, the 6 points may be chosen in C_6^{2n+1} ways.

Therefore, there are, all together,

$$C_3^{2n+1} + 4C_4^{2n+1} + 5C_5^{2n+1} + C_6^{2n+1} = n(2n-1)(2n+1)(2n^3 + 21n^2 - 2n + 9)/90$$

different triangles.

Also solved by Aaron Buchman, Roger Lessard, and the proposer.

Tetrahedron and Concurrent Cevians

E 928 [1950, 483]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a tetrahedron $ABCD$ and a point O. Denote by A', B', C', D' the intersections of AO, BO, CO, DO with the corresponding faces of the tetrahedron, and set $x = AO/A'O$, $y = BO/B'O$, $z = CO/C'O$, $t = DO/D'O$. Show that

$$xyzt = 3 - 2(x + y + z + t) + (xy + xz + xt + yz + yt + zt).$$

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. The altitude of $OBCD$ from O is easily shown to be $h_A/(1-x)$, where h_A is the altitude of $ABCD$ from A. Therefore, if V is the volume of $ABCD$ and b_A is the area of the face opposite A,

$$V = \Sigma(h_A b_A)/3(1-x) = \Sigma V/(1-x).$$

That is

$$1 = 1/(1-x) + 1/(1-y) + 1/(1-z) + 1/(1-t),$$

or

$$xyzt = 3 - 2(x + y + z + t) + (xy + xz + xt + yz + yt + zt).$$

Also solved by Joseph Langr, Roger Lessard and G. S. Mahajani.

A Product of Reflections

E 929 [1950, 483]. *Proposed by H. S. Shapiro, Massachusetts Institute of Technology*

Given three non-concurrent straight lines l_1, l_2, l_3 in the plane. Let T_i denote reflection in l_i and set $T = T_1T_2T_3$. Show that T^2 is a translation.

Solution by B. D. Roberts, New Mexico Highlands University. The double reflection T_1T_2 in the two lines l_1, l_2 (whether intersecting or not) is a rigid sense-preserving transformation involving a rotation through twice the angle from l_1 to l_2. But T^2 is the product of three such double reflections, the sum of the angles involved being a multiple of π. The total rotation due to T^2 is then through an angle which is a multiple of 2π. This means that T^2 produces a translation only. It is easy to show that this translation reduces to the identity transformation if and only if l_1, l_2, l_3 are concurrent or parallel.

Also solved by J. M. Feld, H. C. Kranzer, W. J. Nemerever, C. S. Ogilvy, O. Dale Smith, and the proposer.

Feld used the known fact that the product of three reflections is equivalent to the commutative product of a reflection L in a line l and a translation A parallel to l. Then $T^2 = LALA = ALLA = A^2$, a translation.

Editorial Note. It can be shown that the magnitude of the translation T^2 is equal to twice the perimeter of the orthic triangle of the triangle formed by l_1, l_2, l_3, where, if the latter triangle is obtuse, the side of the orthic triangle corresponding to the obtuse angle is taken as negative. Thus the magnitude of the translation is independent of the order of reflections in the set T. See, *e.g.*, Morley and Morley, *Inversive Geometry*, p. 35.

An interesting consequence of the problem, pointed out by the proposer, is that a plane point set with three non-concurrent and non-parallel axes of symmetry contains an infinite lattice-work of points.

Inverse Tangents

E 930 [1950, 483]. *Proposed by Gordon Raisbeck, Bell Telephone Laboratories.*

If

$$\prod_{i=1}^{n} (x + r_i) \equiv \sum_{j=0}^{n} a_j x^{n-j},$$

show that

$$\sum_{i=1}^{n} \tan^{-1} r_i = \tan^{-1} \frac{a_1 - a_3 + a_5 - \cdots}{a_0 - a_2 + a_4 - \cdots}$$

and

$$\sum_{i=1}^{n} \tanh^{-1} r_i = \tanh^{-1} \frac{a_1 + a_3 + a_5 + \cdots}{a_0 + a_2 + a_4 + \cdots}.$$

Solution by T. M. Apostol, California Institute of Technology. Let p_k be the kth elementary symmetric function of the n variables $r_1, r_2, \cdots, r_n$. Then the formula

$$\tan (x_1 + x_2) = (\tan x_1 + \tan x_2)/(1 - \tan x_1 \tan x_2)$$

is merely the case $n = 2$ of the more general formula

$$\tan\left(\sum_{i=1}^{n} x_i\right) = (p_1 - p_3 + p_5 - \cdots)/(1 - p_2 + p_4 - p_6 + \cdots), \tag{1}$$

where $x_i = \tan^{-1} r_i$. Equation (1) can be proved by induction.

Expressing the coefficients of the polynomial

$$\prod_{i=1}^{n} (x + r_i) = \sum_{j=0}^{n} a_j x^{n-j}$$

in terms of the symmetric functions p_k and substituting in (1) gives the desired result at once.

In the case of the hyperbolic tangent, all the signs on the right of (1) are +.

Also solved by Vern Hoggatt, M. S. Klamkin, Roger Lessard, H. F. Mattson, Azriel Rosenfeld, M. R. Spiegel, O. E. Stanaitis, and the proposer. The proposer used the identities

$$\tan^{-1} a = (-i/2) \log [(1 + ia)/(1 - ia)],$$
$$\tanh^{-1} a = (1/2) \log [(1 + a)/(1 - a)].$$

Drawing Balls from an Urn

E 931 [1950, 556]. *Proposed by H. D. Larsen, Albion College*

Ten balls numbered from 0 to 9 inclusive are placed in an urn. Five of the balls are then drawn at random (without replacement) and arranged in a row. What is the probability that the number thus formed is divisible by 396?

Solution by C. W. Trigg, Los Angeles City College. Since N $(=abcde)$ is divisible be 396 $(=4 \cdot 9 \cdot 11)$ it follows that $S=a+b+c+d+e \equiv 0 \pmod 9$, so $S=18$ or 27. Also, $(a+c+e)-(b+d) \equiv 0 \pmod{11}$. Now if $S=18$, then $b+d=9$, so (b, d) or $(d, b)=(0, 9)$, (1, 8), (2, 7), (3, 6), or (4, 5). There are ten sets of five distinct digits with $S=18$ which contain one of the eligible pairs (b, d) and at least one additional even digit, so that the possibility exists that the terminal pair of a permutation of the set may be divisible by 4. From these we find 64 permutations that are divisible by 396.

If $S=27$, $b+d=8$, so (b, d) or $(d, b)=(0, 8)$, (1, 7), (2, 6), or (3, 5). From the eight sets of five distinct digits which meet these restrictions we find 32 values of N. Therefore the probability that N is divisible by 396 is 96/(10!/5!) or 1/315.

The same result may be obtained by listing all the multiples of 396 between 01188 and 99792 (easily done since $396=400-4$) and deleting those containing duplicate digits.

If k balls are drawn at random from the urn, the probability p that the number found is divisible by 396 may be found by using the divisibility criteria as above. These probabilities are:

k	P
3	2/(10!/7!) = 1/490
4	18/(10!/6!) = 1/280
6	360/(10!/4!) = 1/420
7	1488/(10!/3!) = 31/12600
8	5328/(10!/2!) = 37/12600
9	15984/(10!) = 37/8400
10	78336/(10!) = 34/1575

Also solved by Ferrel Atkins, Monte Dernham, J. B. Friedman, Vern Hoggatt, Roger Lessard, Leo Moser, L. A. Ringenberg, and the proposer.

Triangle with Orthogonal Circum- and Nine Point Circles

E 933 [1950, 557]. *Proposed by Joseph Langr, Prague, Czechoslovakia*

Let A', B', C' be the feet of the altitudes of a triangle ABC whose circumcircle and nine point circle are orthogonal. Show that (1) the areas of triangles ABC and $A'B'C'$ are equal, (2) $(AC')(BA')(CB') = (AB)(BC)(CA)/2$. Construct the triangle ABC given side BC and the position of A' on BC.

Solution by A. Sisk, Maryville, Tenn. Let points A, B, C have coordinates $(0, a)$, $(b, 0)$, $(c, 0)$ respectively. Then A' is at the origin and the coordinates of B' and C' are

$$(c(a^2 + bc)/(a^2 + c^2),\ ac(c - b)/(a^2 + c^2)),$$

$$(b(a^2 + bc)/(a^2 + b^2),\ ab(b - c)/(a^2 + b^2)).$$

The equations of the circumcircle (O) and the nine point circle (N) are found to be

$$x^2 + y^2 - (b + c)x - (a^2 + bc)y/a + bc = 0$$

and

$$x^2 + y^2 - (b + c)x/2 + (bc - a^2)y/2a = 0.$$

The condition that (O) and (N) be orthogonal is

$$a^4 + a^2(b - c)^2 - b^2c^2 = 0. \tag{A}$$

The area ABC is $a(c-b)/2$, and the area of $A'B'C'$ is

$$abc(b - c)(a^2 + bc)/(a^2 + b^2)(a^2 + c^2),$$

which, by use of (A), reduces to $a(b-c)/2$. This proves the first part.

$$(AB)^2(BC)^2(CA)^2/4 = (AC')^2(BA')^2(CB')^2 = bc(b - c)^2(a^2 + bc)/2,$$

and the second part is established.

As for the third part, when we are given BC and the position of A' on BC, we are given b and c. Therefore, by (A), a can be found and the triangle constructed.

Also solved by the proposer, who pointed out that the triangle with orthogonal circum- and nine point circles was studied by V. Thébault in *Mathesis*, tome LVIII, 1949, Question 3409, p. 102 and p. 363. If H is the orthocenter, R the circumradius, and a, b, c the sides of such a triangle, Thébault found that

$$(HA)(HB)(HC) = 4R^2 \quad \text{and} \quad (HA)^2(HB)^2(HC)^2 = a^6 + b^6 + c^6.$$

To these relations the proposer added

$$\cos A \cos B \cos C = -1/2 \quad \text{and} \quad \sin^2 A + \sin^2 B + \sin^2 C = 1.$$

Curves with the "Focal" Property

E 934 [1950, 557]. *Proposed by C. O. Oakley, Haverford, College*

Let C_1 and C_2 be two arbitrarily given curves whose parametric equations are $x_1=x_1(t)$, $y_1=y_1(t)$ and $x_2=x_2(t)$, $y_2=y_2(t)$. Show that there exists a third curve C, $x=x(t)$, $y=y(t)$, with the "focal" property that, if P_1, P_2, and P are corresponding points on C_1, C_2, and C respectively, then P_1P and P_2P make equal angles with the tangent to C at P.

Solution by the Proposer. To prove this we apply the "tangent of the angle between two lines theorem" from analytic geometry. In our notation

$$\left(\frac{y-y_1}{x-x_1}-\frac{y'}{x'}\right)\Big/\left(1+\frac{y-y_1}{x-x_1}\,\frac{y'}{x'}\right)=\left(\frac{y'}{x'}-\frac{y-y_2}{x-x_2}\right)\Big/\left(1+\frac{y-y_2}{x-x_2}\,\frac{y'}{x'}\right),$$

where the primes indicate differentiation with respect to the parameter t. Reducing this we obtain

$$(1)\qquad \begin{aligned}&(y'^2/x'^2-1)[(x-x_1)(y-y_2)+(x-x_2)(y-y_1)]\\&\qquad+2(y'/x')[(x-x_1)(x-x_2)-(y-y_1)(y-y_2)]=0.\end{aligned}$$

The (arbitrary) function $x(t)$ and a corresponding solution $y(t)$ of the non-linear differential equation (1) will thus yield the parametric equations of a curve C satisfying the conditions of the theorem. In actual practice it would be very difficult to carry out the integration except in the very simplest cases.

If each of the two curves be reduced to a single point, say $x_1=-k$, $y_1=0$ and $x_2=k$, $y_2=0$, respectively, equation (1) reduces to

$$(2)\qquad xy(dy/dx)^2+(x^2-y^2-k^2)dy/dx-xy=0,$$

where we have written dy/dx for y'/x'. Equation (2) is the familiar differential equation of confocal conics, in standard position, and the theory shows that for two fixed points the confocal conics are the only curves enjoying this focal property.

In many instances the theorem above can be strengthened. Consider, as before, the two given curves C_1 and C_2 and let a third curve C, $y=f(x)$, be given in cartesian form. A parametric representation of C, $x=x(t)$, $y=y(t)$, often exists such that the focal property holds. For, in this case, equation (1) becomes

$$(3)\qquad \begin{aligned}&(f'^2-1)[(x-x_1)(f-y_2)+(x-x_2)(f-y_1)]\\&\qquad+2f'[(x-x_1)(x-x_2)-(f-y_1)(f-y_2)]=0,\end{aligned}$$

a non-differential equation in x and t. From a real solution $x=x(t)$ of (3), when such exists, we obtain the parametrization sought since $y=y(t)=f(x(t))$.

Also solved by A. Sisk.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed.

PROBLEMS FOR SOLUTION

4433. *Proposed by J. W. Gaddum, University of Missouri*

Let $f(z)$ be a continuous function of a complex variable, finite at z_0. Then in every neighborhood of z_0 there are distinct z_1, z_2 such that $|f(z_1)| = |f(z_2)|$.

4434. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Prove

$$\sum_{k=1}^{n-1} \frac{\sin^2(kr\pi/n)}{\sin^2(k\pi/2n)} = 2r(n-r), \qquad 0 \leqq r \leqq n,$$

and thus evaluate

$$\int_0^1 \frac{\sin^2 rx}{\sin^2 \frac{1}{2}x}\,dx.$$

4435. *Proposed by J. P. Ballantine, University of Washington, Seattle*

Find the locus of points in xyz-space from which the conic $b^2x^2+a^2y^2=a^2b^2$ in the plane $z=0$ looks like a circle.

4436. *Proposed by D. J. Newman, New York University*

What is the probability that an arbitrary integer have a prime divisor which is larger than its square root?

4437. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Here is an old problem of Turán: Let $z_1, z_2, \cdots, z_n$ be n complex numbers, $|z_1| = 1$. Put $s_k^{(n)} = z_1^k + z_2^k + \cdots + z_n^k$. Does an absolute constant c exist, independent of n, so that

$$\max_{1 \leq k \leq n} \left| s_k^{(n)} \right| = d_n > c?$$

SOLUTIONS

Isosceles Tetrahedron

4354 [1949, 414]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A necessary and sufficient condition for a tetrahedron to be isosceles is that each of two bialtitudes of the tetrahedron divide the opposite edges proportion-

ally.

Solution by L. M. Kelly, Michigan State College. Consider the parallelepiped circumscribed about the tetrahedron $ABCD$. If $ABCD$ is isosceles, the diagonals of the various faces are pairwise equal so that the parallelepiped is rectangular and the bialtitudes clearly bisect the opposite edges.

Conversely, let P and Q be the feet of a bialtitude, P being on AB and Q on CD, and suppose $AP/PB = CQ/QD$. Let the second diagonal in the face with AB be labelled $C'D'$, and locate point Q' on $C'D'$ such that $Q'D' = QD$. Then QQ' is parallel to DD' and $Q'P$ is parallel to $D'B$. Thus the face $DD'B$ is parallel to the plane $QQ'P$ containing the bialtitude and is therefore perpendicular to the face $ABD'C'$.

If another bialtitude divides its opposite edges proportionally then the parallelopiped will be rectangular and the tetrahedron isosceles.

Also solved by the Proposer.

Another Sequence of Remarkable Squares

4355 [1949, 479]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Solve the equation

$$5m^2 + 2m + 1 = n^2$$

in integers and show that, provided $m \equiv -1 \bmod 3$, there exists in every system of numeration of base $B = 3m+1$ at least one pair of perfect squares having the form $aabb = (cc)^2$, $bbaa = (dd)^2$. There are infinitely many such systems of numeration. (See also the Proposer's paper, "Two Classes of Remarkable Perfect Square Pairs," this MONTHLY, 1949, pp. 443–448).

Solution by Daniel Block, Yeshiva College, New York City. Solving $5m^2+2m+1=n^2$ for m, we obtain

$$m = (-1 + k)/5, \qquad k^2 = 5n^2 - 4,$$

or $(k/2)^2-5(n/2)^2=-1$. Since the convergents for the continued fraction of $\sqrt{5}$ are 2/1, 9/4, 38/17. 161/72, 682/305, $\cdots$, we have solutions $k/2=p$, $n/2=q$, where p/q is any convergent of odd order. Thus

$$n = 2, \quad k = -4, \quad m = -1; \qquad n = 34, \quad k = 76, \quad m = 15;$$
$$n = 610, \quad k = -1364, \quad m = -273; \cdots .$$

In order that perfect squares exist as prescribed, there must be integers a, b, c, d which satisfy conditions (1), (2), (3), (4) of the Proposer's paper [1949, 444]. These are obtained with little effort by setting

$$c = n, \qquad d = 2m + 1, \qquad a = (5m + 2)/3, \qquad b = (4m + 4)/3.$$

a and b will be integers by virtue of the condition $m \equiv -1 \bmod 3$.

Also solved by Roger Lessard and by the Proposer.

Editorial Note. It is easily verified that if k_i, n_i are any positive integers satisfying $k^2=5n^2-4$, then k_{i+1}, n_{i+1} will also satisfy if

$$k_{i+1} = 9k_i + 20n_i, \qquad n_{i+1} = 4k_i + 9n_i; \tag{1}$$

furthermore, that k_i, n_i may be obtained from $(k_1, n_1)=(1, 1)$ or $(4, 2)$ or $(11, 5)$ by repeated use of (1). There are therefore three infinite sequences of values of k_i and n_i. A recursion formula for k_i alone, obtained by eliminating the n's, is

$$k_{i+1} = 18k_i - k_{i-1}. \tag{2}$$

To meet the requirement that m and a, as given above, be positive integers we must have $k_i \equiv -4 \bmod 15$. From (2) and the values of (k_1, n_1) it is evident that the values of k_i in each sequence are congruent mod 15 to $1, -1, -4, 4, 1, -1, \cdots$ in cyclic succession. Hence every fourth value in each sequence (starting at the proper member) provides a satisfactory set of squares. The simplest is, to base 7, $4444=(55)^2$. The next is given by $B=313$, $a=174$, $b=140$, $c=233$, $d=209$. The smallest given by the sequence in the above solution has $B=14686$.

Two Summations

4356 [1949, 479]. *Proposed by P. A. Piza, San Juan, Puerto Rico*

Prove the relations

(a) $$x^{2n+1} - (x-1)^{2n+1} = \sum_{a=0}^{n}\left[\binom{n+a}{2a+1} + \binom{n+1+a}{2a+1}\right](x^2-x)^{n-a}$$

(b) $$x^{2n+2} - (x-1)^{2n+2} = (2n-1)\sum_{a=0}^{n}\binom{n+1}{2a+1}(x^2-x)^{n-a}.$$

I. *Solution by Max LeLeiko, Brooklyn, New York.* These formulas are special cases of more general formulas given in Chrystal, *Algebra*. From part II, page 203 we take the identity

$$\alpha^{2n+1} + \beta^{2n+1} = (-1)^n(2n+1)\sum_{a=0}^{n}(-1)^a\frac{\binom{n+a}{2a}}{2a+1}p^{2a+1}q^{n-a},$$

where $p=\alpha+\beta$, $q=\alpha\beta$. Now let $\alpha=x$, $\beta=1-x$. Then $p=1$, $q=x-x^2$, and substituting we get

$$x^{2n+1} + (1-x)^{2n+1} = (2n+1)\sum_{a=0}^{n}(-1)^{n+a}\frac{\binom{n+a}{2a}}{2a+1}(x-x^2)^{n-a}.$$

Using

$$\frac{2n+1}{2a+1}\binom{n+a}{2a} = \binom{n+a}{2a+1} + \binom{n+1+a}{2a+1}$$

we find proposed relation (*a*).

To prove (b) we make use of another identity (page 204 of the same reference):

$$\alpha^{2m} - \beta^{2m} = (\alpha - \beta)(-1)^{m-1}\sum_{a=0}^{m-1}(-1)^a\binom{m+a}{2a+1}p^{2a+1}q^{m-a-1}.$$

With $\alpha = x$, $\beta = 1-x$, $m = n+1$, $p = 1$, $q = x-x^2$, this is easily reduced to the desired relation.

II. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York.* Assume (b) is true for $n = k$. Then by integrating between 1 and x we obtain

$$x^{2k+3} - (x-1)^{2k+3} = 1 + \sum_{a=0}^{k}\frac{2k+3}{k+1-a}\binom{k+1+a}{2a+1}(x^2-x)^{k+1-a}$$

which is easily shown to be equivalent to (a) for $n = k+1$. Further, upon multiplying (a) for $n = k+1$ by $2x-1$, multiplying (b) for $n = k$ by x^2-x, and subtracting, we get exactly (b) for $n = k+1$. Since (a) and (b) are obviously true for $n = 0$, it follows by induction that they are true for all integral n.

Also solved by Roger Lessard.

A Summation Equivalent to an Integration

4357 [1949, 479]. *Proposed by R. D. Stalley, University of Arizona*

Reduce the problem of summing the series

$$\sum_{k=1}^{\infty} k^{-n}x^k$$

where n is a positive integer $\geqq 2$ to numerical integration of a function over a finite range.

Solution by the Proposer. Assume $|x| < 1$. From the relation

$$x\frac{d}{dx}\sum_{k=1}^{\infty}k^{-n}x^k = \sum_{k=1}^{\infty}k^{-n+1}x^k,$$

we have

$$\sum_{k=1}^{\infty}k^{-n}x^k = \int_0^x \frac{1}{x}\sum_{k=1}^{\infty}k^{-n+1}x^k dx,$$

the constant of integration being 0. If $n = 1$, this equation yields

$$\sum_{k=1}^{\infty}k^{-1}x^k = \int_0^x \frac{1}{x}\sum_{k=1}^{\infty}x^k dx = \int_0^x \frac{1}{x}\cdot\frac{x}{1-x}dx = \log\frac{1}{1-x}.$$

If $n = 2$, we have

$$\sum_{k=1}^{\infty} k^{-2}x^k = \int_0^x \frac{1}{x} \log \frac{1}{1-x}\, dx.$$

Repeating this process for successive integers we see inductively that

$$\sum_{k=1}^{\infty} k^{-n}x^k = \int_0^x \frac{1}{x}\int_0^x \frac{1}{x} \cdots \int_0^x \frac{1}{x} \log \frac{1}{1-x}\, (dx)^{n-1}.$$

Round-the-World Flight

4360 [1949, 556]. *Proposed by Free Jamison, San Jose State College, California*

Any one of a group of airplanes may be refueled from any other. Each has a fuel capacity sufficient for a flight one-fifth the distance around the earth. Assuming that all have the same constant ground speed and the same rate of fuel consumption, that the only landing place and the only available fuel supply are at the home base, and that refueling time is negligible, find the minimum number of planes necessary so that one plane may fly around the earth and all return home safely. (Compare *The Jeep Problem* [1947, 24], and [1947, 458].)

Comments by the Proposer. The minimum number of planes required is not more than 75.

It seems reasonable that a sufficient number of planes would depart from the home base so that, by refueling, one plane would be fueled to capacity when two-fifths of the distance around the earth and be met there by a plane flying in the opposite direction. Let two-fifths of the distance be divided into twelve equal legs. (Each leg is one-thirtieth of the earth's circumference, a plane fueled to capacity has enough fuel for six legs.) A schedule is given to show how 77 planes can accomplish the desired feat. Thirty-two planes depart at the same time, at the end of the first leg 25 are fueled to capacity and 7 return, at the end of the second leg 5 turn back after refueling 20 to capacity but at the same instant 9 depart from the home base, and so on. Let the amount of fuel necessary for one plane for one leg be called a portion. Subscripts on numbers of returning planes indicate portions of fuel remaining at the end of the leg, negative subscripts on numbers of outbound planes indicate portions of fuel below capacity at the beginning of the leg.

After six legs have been completed by the first flight of planes it would be necessary to send planes from the home base in the opposite direction to meet the plane flying around the earth. For this operation we have only to use the lines of the schedule in reverse order. The numbers of planes in the air at various times are now easily computed, and it is found that no more than 77 are required at any one time. However this number can be reduced by one if on the eighth outbound flight of ten planes, one plane turns back at the midpoint of the first leg after refueling the other nine planes (with a similar change in the analogous flight which is to meet the home-coming plane). A further reduction of one is possible if on the sixth, seventh, ninth and tenth flights each, one plane is turned back at the midpoint of the first leg; and if on the eight flight

one plane is turned back at the quarter point and one at the three-quarter point on the first leg.

It is reasonable to suppose that the minimum number demanded in the proposal is less than 75, since the schedule here presented makes the simplifying assumptions that changes of direction and refueling occur only when an integral number of legs have been traversed and that all planes flying at any instant are flying in the same direction (indicated by the arrow.)

Somewhat similar comments were received from W. E. Buker and J. V. Pennington.

$$
\begin{array}{cccccccccccccc}
32 & & & & & & & & & & & & & \\
\rightarrow & & & & & & & & & & & & & \\
7 & 25 & & & & & & & & & & & & \\
\leftarrow & \rightarrow & & & & & & & & & & & & \\
9 & 5_0 & 20 & & & & & & & & & & & \\
\rightarrow & \leftarrow & \rightarrow & & & & & & & & & & & \\
8 & 6 & 4_0 & 16 & & & & & & & & & & \\
\leftarrow & \rightarrow & \leftarrow & \rightarrow & & & & & & & & & & \\
18 & 6_0 & 4 & 4_4 & 12 & & & & & & & & & \\
\rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & & & & & & & & & \\
11 & 13 & 5_1 & 3 & 3_3 & 9 & & & & & & & & \\
\leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & & & & & & & & \\
13 & 9_3 & 9 & 4_2 & 2 & 2_1 & 7 & & & & & & & \\
\rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & & & & & & & \\
13 & 9 & 7_4 & 6 & 3_2 & 1 & 2_3 & 5 & & & & & & \\
\leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & & & & & & \\
15 & 10_3 & 6 & 5_3 & 4 & 2_1 & 1_{-1} & 1_0 & 4 & & & & & \\
\rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & & & & & \\
15 & 10 & 7_2 & 4 & 3_0 & 3 & 2_2 & & 1_1 & 3 & & & & \\
\leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & & & & \\
14 & 10_0 & 7 & 5_3 & 2 & 3_2 & 2 & 1_0 & & 1_2 & 2 & & & \\
\rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & & & \\
15 & 9 & 7_1 & 5 & 4_2 & 1 & 1_0 & 2_{-3} & 1_1 & & & 2_{-2} & & \\
\leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & & \\
12 & 10_0 & 6 & 6_3 & 3 & 1_0 & 1_{-2} & 2_0 & 1 & & & 1_1 & 1 & \\
\rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \\
15 & 7 & 8_1 & 4 & 2_1 & 2 & 3_0 & & & 1_{-1} & 1_0 & & & \\
\leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & & & \\
10 & 11_1 & 4 & 3_0 & 3 & 4_0 & 1 & & & 2_2 & & & & \\
\rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & & & & \\
15 & 6 & 5_3 & 2 & 6_3 & 1 & & 1_{-1} & 2_0 & & & & & \\
\leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & & & & & \\
9 & 7_2 & 4 & 7_0 & 1 & & 1_{-1} & 3_1 & & & & & & \\
\rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & & & & & & \\
10 & 6 & 9_0 & 2_{-1} & & 1_{-1} & 4_1 & & & & & & & \\
\leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & & & & & & &
\end{array}
$$

$$\begin{array}{cccccc} 7 & 12_0 & 3 & & 2_{-3} & 5_0 \\ \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow \end{array}$$

$$\begin{array}{cccccc} 16 & 3 & 1_2 & 2 & 7_0 \\ \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow \end{array}$$

$$\begin{array}{cccc} 4 & 2_3 & 2 & 9_1 \\ \rightarrow & \leftarrow & \rightarrow & \leftarrow \end{array}$$

$$\begin{array}{ccc} 3 & 3 & 11_0 \\ \leftarrow & \rightarrow & \leftarrow \end{array}$$

$$\begin{array}{cc} 4 & 14_1 \\ \rightarrow & \leftarrow \end{array}$$

$$\begin{array}{c} 18 \\ \leftarrow \end{array}$$

Units of time are indicated vertically, distances (in legs) horizontally.

An Application of Helly's Theorem

4361 [1949, 557]. *Proposed by Melvin Dresher, The Rand Corporation.*

If $S_1, S_2, \cdots, S_m$ are m line segments parallel to the y-axis such that through every set of $n+2$ of them the locus of an nth degree polynomial can be passed, then there exists some nth degree polynomial whose locus intersects all m segments.

I. *Solution by Robert Steinberg, University of California, Los Angeles.* The theorem is true for $m=n+2$. Hence, we take $m>n+2$ and assume that the theorem is true for $m-1$ as a basis for an inductive proof.

Let f_1 be the polynomial through the segments $S_2, S_3, \cdots, S_m$, and let $f_2, f_3, \cdots, f_m$ be similarly defined. Consider the equations:

$$\sum_{i=1}^{m} a_i f_i = 0, \qquad \sum_{i=1}^{m} a_i = 0,$$

where the first equation is considered to be an identity in x. These give $n+2$ linear homogeneous equations to determine m a_i's, with $m>n+2$, and hence there is a solution in which the a_i's are not all zero. Let all the positive a_i's be included in $a_1, a_2, \cdots, a_r$, and the negative a_i's in $a_{r+1}, \cdots, a_m$.

Then, the polynomial

$$f = \frac{\sum_{i=1}^{r} a_i f_i}{\sum_{i=1}^{r} a_i} = \frac{\sum_{i=r+1}^{r} (-a_i) f_i}{\sum_{i=r+1}^{r} (-a_i)}$$

is a polynomial which intersects all m segments. For, from its two representations, it is obviously intermediate to $f_1, f_2, \cdots, f_m$ for any value of x; and, from the first representation, it is seen to intersect $S_{r+1}, \cdots, S_m$ and, from the second, $S_1, S_2, \cdots, S_r$.

This completes the proof.

II. *Solution by S. H. Gould, Purdue University.* Let S_k have the end-points (x_k, p_k) and (x_k, q_k) with $p_k < q_k$. Interpret the coefficients a_i of the polynomial

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n$$

as coördinates of a point A in $(n+1)$-Euclidean space. Then the set of points A for which

$$p_k \leqq f(x_k) \leqq q_k \tag{1}$$

is convex; for, if $a_i'' = (1-\lambda)a_i + \lambda a_i'$, with $0 \leqq \lambda \leqq 1$, and if A and A' satisfy (1), then clearly A'' also satisfies (1).

Thus, the problem becomes a particular case of Helly's Theorem (see e.g. Alexandroff-Hopf, *Topologie*, p. 297): *in order that any finite number m of convex point-sets in $(n+1)$-Euclidean space shall have a point in common, it is necessary and sufficient that every set of $(n+2)$ of them shall have a point in common.* Simple proofs were given by Radon (*Math. Ann.* v. 83, 1921, p. 113), Konig (*Math. Zeitschr.* v. 14, 1922, p. 203), and Helly (*Jber. Deutsch. Math.-Vereinig.* v. 32, 1923, p. 175).

Evaluation of Determinant

4362 [1949, 557]. *Proposed by D. J. Newman, New York University*

Evaluate

$$\begin{vmatrix} 1 & 1/2 & 1/3 & \cdots & 1/n \\ 1/2 & 1/3 & 1/4 & \cdots & 1/(n+1) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1/n & 1/(n+1) & 1/(n+2) & \cdots & 1/(2n-1) \end{vmatrix}.$$

I. *Solution by N. D. Lane, Acadia University, Nova Scotia.*

Subtract the nth column from the 1st, 2nd, $\cdots$, $(n-1)$th columns in turn, and factor out the common factors of the rows and of the columns. Now subtract the nth row from the 1st, 2nd, $\cdots$, $(n-1)$th rows in turn and again factor out the common factors of the rows and of the columns. This gives the recurrence formula

$$F(n) = \frac{[(n-1)!]^4}{(2n-1)!(2n-2)!} \cdot F(n-1),$$

where $F(n)$ is the given determinant. Thus we obtain

$$F(n) = \frac{[1!2!3! \cdots (n-1)!]^4}{1!2!3! \cdots (2n-1)!}.$$

The same procedure will give the evaluation of a determinant whose ele-

ments a_{ij} are the reciprocals of the $(i+j-1)$th terms of any arithmetic progression. The determinant has the value

$$F(n) = [1!2!3! \cdots (n-1)!]^2 d^{n(n-1)} \prod,$$

where $\prod$ is the product of all the elements of the determinant.

II. *Solution by P. W. M. John, University of Oklahoma.* The following proof employs an idea suggested by A. C. Aitken. Consider the determinant $|A|$, (the Cauchy double alternant)

$$|A| = |(a_i - b_j)^{-1}|.$$

Clearing the fractions by multiplying each row by the continued product of the denominators of that row, we have

$$|A| = \frac{1}{\prod_{i,j} (a_i - b_j)} |C|$$

where the (ij)th element of C is $[\prod_j (a_i - b_j)]/(a_i - b_j)$. Then C is a polynomial of order $n(n-1)$ in the a_i, b_j. It vanishes when $a_i = a_j$ and when $b_i = b_j$ $(i \neq j)$. Hence it is divisible by the two difference products, $\Delta(a_1, a_2, \cdots, a_n)$ and $\Delta(b_1, b_2, \cdots, b_n)$. Since their combined degree is $n(n-1)$, the remaining factor is numerical. To find it, put $a_i = b_i$, when all terms except those in the leading diagonal vanish. The continued product of the diagonal elements gives the difference product twice. However, with each factor $(a_i - a_j)$ occurs also $(a_j - a_i)$, so that the numerical term is seen to be $(-1)^{n(n-1)/2}$. Then

$$|A| = \frac{|C|}{\prod_{i,j} (a_i - b_j)} = \frac{(-1)^{n(n-1)/2} \Delta(a_1, \cdots, a_n) \Delta(b_1, \cdots, b_n)}{\prod_{i,j} (a_i - b_j)}.$$

The value of the proposed determinant may be obtained easily by setting $a_i = n+1, n+2, \cdots, 2n$ and $b_j = n, n-1, \cdots, 2, 1$.

Also solved by A. C. Aitken, F. Bagemihl, Sol Ciolkowski, A. R. Erskine, Daniel Finkel, H. L. Krall, Emma Lehmer, Julius Lieblein, C. D. Olds, E. G. Olds, Ingram Olkin, R. H. Pennington, D. W. and J. R. Pounder, H. D. Ruderman, W. Seidel, J. H. Simester, O. E. Stanaitis, Robert Steinberg, H. E. Stelson, and the Proposer.

The following references were noted:

Bourbaki, N., Algebra, p. 90, ex. 5
Cauchy, Œuvres complètes, 2 serie, v. xii, p. 177.
Hilbert, D, Ein Beitrag zur Theorie des Legendreschen Polynoms, Gesammelte Abhandlungen, v. 2, p. 369.
Ligowski, Archiv. d. Math. u. Phys. v. xxxvi, 1861, pp. 181–185.
Muir, T., The Theory of Determinants, III, 1920, p. 311.
Muir and Metzler, The Theory of Determinants, 1930, pp. 429–431.
Schrader. W., Beiträge zur Theorie der Determinanten, Halle, 1887.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Elements of Analytical Geometry. By W. L. Hart. D. C. Heath and Company, Boston, 1950. 264 pages. $2.75.

This book is a new addition to W. L. Hart's list of carefully written texts for freshman mathematics. Those teachers of freshman mathematics who have enjoyed his *College Algebra* and his *Trigonometry*, will welcome this book into the field of available texts in analytical geometry.

This book cannot be cited for any radical departures from the customary content of the course in analytical geometry as taught in most colleges and universities in the United States. It gives an adequate treatment of a minimum course in plane analytical geometry, followed by a treatment of solid analytical geometry, which should meet the needs of most students of first year calculus. The last chapter of the book, considers, in an elementary way, the problem of fitting a curve to empirical data.

The author has given considerable attention to the need for carefully defined mathematical terms. He has also seen the need for providing a summarizing chapter on the function concept and the definition of terms usually encountered in freshman mathematics when studying this topic. A short chapter, (Chapter 2) entitled, "Related Variables," presents the minimum content pertaining to this topic.

It is becoming increasingly popular to present the subject-matter of freshman mathematics in as unified a manner as possible. This objective has not been neglected by the author. In his own words, "As a unifying aim, the book endeavors to present a foundation for the joint use of analytic methods and graphical representation, oriented with respect to their later applications. In particular, this viewpoint induces emphasis on fundamental terminology relating to variables, functions and their graphs, and informal but accurate use of the language of limits on a few occasions." However, in the opinion of the reviewer, this book, and almost all other books on analytic geometry, could achieve greater unity if the authors had made more use of such ideas as direction cosines in plane analytical geometry thereby more readily leading to the study of lines and planes in space and achieving an even greater unity than can be achieved by introducing such ideas very late in the course.

HENRY VAN ENGEN

Giant Brains, or Machines That Think. By C. Berkeley. John Wiley and Sons, Inc., New York, 1949. 7+270 pages. $4.00.

Giant Brains, or Machines That Think is a semi-popular book about large scale computing machines. The book begins by discussing the provocative subtitle, "machines that think." While the author makes a good case (and an even better one can now be made), it seems to the reviewer that the definition of "think" will usually be framed, humans being ego-centric, to exclude machines, much as "soul" is usually defined to exclude animals.

For the beginner in the field, the author devotes a chapter and two appendices to the language of computing machines. Chapter 3 describes a simple machine, called Simple Simon, designed by the author to illustrate the nature of digital computing machines, and as such, the machine is an excellent example.

The bulk of the book is devoted to descriptions of the major computing machines which were completed and in use during the years 1942–6. These machines include IBM accounting equipment, the MIT Differential Analyser (No. 2), the Harvard Automatic Sequence-Controlled Calculator (Mark I), the Moore School ENIAC (now at Aberdeen Proving Grounds), the Bell Laboratories' General-Purpose Relay Calculators (Models 5), and the Kalin-Burkhart Logical-Truth Calculator. One chapter is devoted to each of these machines, and in each case, the author shows in simple colorful language the main features of the machine. Each of these chapters is a mine of information, and collectively (along with a very excellent set of references in the back of the book) they form a valuable contribution to the literature of large scale computing machines.

An additional chapter is devoted to recent developments in the art of designing and building these machines. The chapter includes a brief description of the Harvard Mark II, the IBM S-SEC, and the Eckert-Mauchly BINAC.

In the next to the last chapter, the author extrapolates from recent developments in all directions—many of which extrapolations are provocative and some of which may well be true! On the other hand, the last chapter on Social Control might well have been left out.

In conclusion, it is only fair to say that the book represents an enormous labor of love on the part of the author, and that in spite of its elementary style of writing and occasional technical inaccuracies, it will be useful to the beginner in the field as well as to the professional.

R. W. HAMMING

Fourier Methods. By Philip Franklin. McGraw Hill, New York, 1949. 10+289 pages. $4.00.

This is a book for engineers and other technicians. It is designed to show them how to operate with complex quantities and how to solve problems which depend for their solution on the use of Fourier series and integrals, and Laplace transforms. It is vigorous mathematics at its extreme; the emphasis is not on the concepts involved, but on how to use these concepts to work problems.

The author makes no attempt to lean on the mathematical rigor, but nevertheless the analysis is done in orderly fashion; the arguments are clear and plausible, and important facts, such as those concerning the convergence of Fourier series are plainly stated without claiming to be proved.

The book consists of five large chapters. The first is devoted to manipulation of complex numbers and of the elementary transcendental functions of a complex variable. Complex power series and the notion of the derivative of a complex function are used.

The second chapter is devoted mainly to Fourier series, although about five pages near its end are used to define and describe Fourier and Laplace transforms. The author states in his introduction that the approach to Fourier series is through averages and root mean square values; by this is meant that he regards the process of finding the coefficients as getting the averages of certain trigonometric products. The notions of orthogonality and least square fit are not at all exploited; as far as the reviewer can tell, the words do not appear in the book. It is probably the book's weakest feature that these ideas, which seem so essential for the understanding of Fourier series, and which unite so many superficially different ideas and processes in mathematical analysis, are completely ignored. When the harmonic analysis of a function given numerically is taken up, it is considered as a problem in finite curve fitting, although the notion of curve fitting is used nowhere else, and in spite of the fact that the formulas obtained for the coefficients in this way are essentially the usual Fourier formulas, with the integrals replaced by Riemann sums. The treatment of the process of expanding a function is very down to earth; various kinds of symmetry are treated in detail, and all discussions are in terms of functions of period p instead of 2π. For this latter to be done throughout a large work, unless it is a reference manual, seems to the reviewer to be a waste of typography and a confusion to the eye. An engineer who really has difficulty in managing a linear transformation ought have no business operating with anything as gaudy as a Laplace transform.

Chapter III is devoted to the derivation and/or description of a large number of partial differential equations, mainly those of mathematical physics, and to some elementary methods of solution-partial integration, operator factorization, *etc.* Certain parts of the chapter use vector analysis considerably, in particular and of necessity the part dealing with Maxwell's equations. Chapter IV contains the solutions of many of these equations, together with the appropriate boundary conditions, in terms of Fourier series. Problems considered include heat flow, vibrating string, power transmission, and wave guides.

In Chapter V, which is the largest chapter in the book, is found the application of the Laplace transform to the solution of ordinary and partial differential equations. There is a small table of transforms and a considerable discussion of partial fractions, with what amounts to residue computation without being called such. The main applications are to electrical networks and power transmission problems.

There is an extremely large collection of problems. The problem set on the computation of Laplace transforms, for example, contains sixty-four problems, ranging from straightforward definite integrations to significant theoretical results requiring considerable epsilon and delta analysis.

In his introduction, the author states that the student should have a working knowledge of elementary calculus, and that most of the topics can be covered in one semester. The reviewer is of the opinion that it would be a very busy semester.

J. W. GREEN

Gelöste und Ungelöste Mathematische Probleme aus Alter und Neuer Zeit. By Heinrich Tietze (Professor of Mathematics, University of Munich, Bavaria, Germany). Biederstein Publishers, Munich, 1949. Volume I: xx, 256 pp., 8°, 115 figs., 10 plates; Volume II: iv, 305 pp., 8°, 41 figs., 8 plates. Price: Paper Bound 18 DM, Linen Bound 25 DM (German Marks).

These two volumes are based on a series of lectures given repeatedly to students of all curricula, at the University of Munich, by the author. They are intended for readers interested in mathematical problems, as suggested by the subtitle. Although the author states specifically that the work is not a treatise on mathematics, the student and advanced reader will find an abundance of information in the "Remarks" which are gathered at the end of each volume.

The following chapter titles serve to show the coverage: Prime Numbers and Prime Twins; Journeys on Surfaces (Geodesy); Trisection of Angles; Adjoining Regions (Topology); Squaring the Circle; Three and More Dimensions; More About Prime Numbers; Counts and Calculations; the Regular 17-Gon; Solution of Algebraic Equations by Radicals; The Four-Color Problem; The Infinite in Mathematics; Fermat's Last Theorem; Curvature of Space.

A considerable amount of effort is devoted to the thorough development of each problem. This is done with the same eloquence to which Professor Tietze's students are accustomed. Numerous biographies of great mathematicians and their portraits provide a rich historical setting for the understanding of the growth of the problems. Additional illustrations are contained in each chapter; the color plates for the Möbius strips and for the Four-Color Problem are extremely well designed and harmonious in the choice of their hues. The tabulations, special problems, detailed proofs, historical notes, literature, bibliography, *etc.*, which comprise the "Remarks" are a veritable fountain of information. This work will prove to be a prized possession in any mathematical library.

CARL HAMMER

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1949–50

Pi Mu Epsilon, Carnegie Institute of Technology

The *Pennsylvania Epsilon* chapter of *Pi Mu Epsilon* reports that, for the academic year 1949–50, six meetings were held, including a business meeting and the annual initiation and banquet. The following papers were presented:

The so-called crisis in the foundations of mathematics and logic, by Prof. Abraham Fraenkel of the Hebrew University at Jerusalem

The mathematical theory of games, by Prof. H. A. Simon

Four-space representations of functions of a complex variable, by Prof. James Taylor of the University of Pittsburgh

Applications of electronic computing equipment to technical and scientific problems, by C. C. Hurd of the International Business Machines Corporation

Mathematical and inductive proofs, by Visiting Professor Albert Schaeffer, formerly of Purdue University and now of the University of Wisconsin.

The initiation of new members and the annual banquet of the chapter were held on April 26, 1950. Fifty-seven persons were initiated, of which twenty-three were undergraduates, twenty were graduates, and fourteen were faculty members.

The officers for 1949–50 were: Director, Richard DiPrima; Vice-Director, Douglas Shaffer; Secretary, William Warner; Treasurer, F. B. Smith, Jr.; Faculty Adviser, Prof. J. B. Rosenbach; Executive Committee, Richard Cutkosky, Robert McKelvey, and William Simon.

Mathematics Club, Hunter College

The *Mathematics Club* of Hunter College held bi-weekly meetings during the Spring semester of 1950. The following talks were presented:

The lighter side of mathematics, by Marie Hyman

Mathematical treasures at Hunter College, by Carolyn Eisele

Hyperbolic functions, by Marjorie Hull and Jerry Greyson

Some topological ideas, by Marion Walter

Finite geometry, by Prof. Jewell H. Bushey.

In addition to the above, Mr. Rutherford Boyd gave an illustrated lecture on *Mathematical forms in art*. His film, *Parabola*, was also shown. A few weeks later, members of the Club and Faculty spent a delightful afternoon at his home learning more about his mathematical art—or artistic mathematics.

During the term a dinner, a theatre party, and a picnic were also enjoyed by the members.

The officers for the semester were: President, Marion Walter; Vice-President,

Dorothy Freudenberger; Secretary, Anita Friedman; Treasurer, Evelyn Hovrath; Faculty Advisor, Prof. Anita Tuller.

Kappa Mu Epsilon, Pomona College

Papers presented to the *California Alpha* chapter of *Kappa Mu Epsilon* during the academic year 1949–50 were:

Method of casting out nines, by Donald Benson

Sir Isaac Newton, by Carolyn Grove

Linkages, by John Dienes

Basic problems of number theory, by Ralph Vernon

Conic sections, with illustrations in plastic, by Dr. C. G. Jaeger and Mr. Charles Halberg

Navigation, by Gilbert Madden

Continued fractions, by Donald Benson

Polygonal numbers, by Dr. Jean Walton

Diophantine equations, by William Paxton

Existence proof for transcendental numbers, by Richard Edelstein

Boolean algebra, by Rodney Weldon

The numbers racket, by Dr. Aubrey Kempner

Rigidity of the earth, by Dr. Walter Whitney.

The annual banquet of the chapter followed the initiation of twenty-three new members. Dr. Aubrey Kempner spoke on *What is truth in mathematics*?

Officers elected for 1950–51 are: President, Walter Rosenow; Vice-President, Rodney Weldon; Secretary-Treasurer, Zoe Lindberg; Social Chairman, Richard Edelstein.

Kappa Mu Epsilon, Central Missouri State College

The *Missouri Beta* chapter of *Kappa Mu Epsilon* held regular monthly meetings during the academic year 1949–50 and the summer session of 1950. The following talks were presented by members of the Society:

Denumerable and non-denumerable infinities, by Sammy Vaughn

The Euclidean algorithm, by Wayne Vanderlinden

Types of discontinuities, by Philip Burford

Nomography, by Rex Wyrick

Complex numbers and their geometric interpretation, by Kathryn Lou Baker

Puzzles based on binary and ternary number systems, by Martin Rowland

Short-cuts in multiplication, by Peggy June Taylor

The fourth dimension, by Barbara Wurth

An oblique coordinate system, by Mrs. Marcia Jackson

Cryptography, by Wilfred Poese

The maneuvering board, by Charles Sigrist

The expression of $\cos n\theta$ *as* $f(n, \cos \theta)$, by Charles Warden

Two theorems on prime numbers, by Michael Stratton

A minimum body of mathematical concepts and information for high school graduates, by Mrs. Margaret Shrake

Volumes and surface areas of regular polyhedra, by Charles Warden.

At the annual banquet in the Spring, Mr. Ray Watson, a sales and market analyst for Hallmark Brothers of Kansas City, Missouri, spoke to the society on *Market potentials—their use and instructions*. The annual Summer picnic was held in May.

Mr. Keith Stumpff was elected President for 1950–51. Interim officers serving during the Summer of 1950 were: Vice-President, Isabella Clarke; Secretary, Mrs. Marcia Jackson; Treasurer, Charles Warden; Faculty Sponsor and Corresponding Secretary, Mr. Loren W. Akers.

Pi Mu Epsilon, Oklahoma Agricultural and Mechanical College

The following discussions were made at the regular meetings of the *Oklahoma Beta* chapter of *Pi Mu Epsilon* during the year 1949–50:

The foundations of mathematics, by Prof. Arnold Dresden of Swarthmore College

The elements of Hilbert space, by Prof. Ainsley Diamond

Non-Euclidean geometry, by Prof. J. V. Robinson

The relation of mathematics to the work of an engineer, by Prof. R. R. Rogosinski, Kings College, Newcastle-upon-Tyne, England

Some aspects of the history of science, by Prof. Orville Schultz of the department of botany.

The chapter had 75 members in residence at the beginning of the year and initiated 18 members. The total membership of the chapter since its installation in 1938 is approximately 350.

Kappa Mu Epsilon, Mount Mary College

The *Wisconsin Alpha* chapter of *Kappa Mu Epsilon* held monthly meetings during the year 1949–50. Besides movies on the solar system and on weather conditions, the following papers were read:

Non-Euclidean geometry, by Joan Daley

The fourth dimension, by Mary Kilkelly

History of the calculus, by Mary Hund

Relativity, by Kathleen Hanley

Number systems, by Wanda Kropp

The slide rule, by Janet Haig

Planetariums, by Betty Prossen.

Dorothy Karner read a paper on *Inversion* at the *Kappa Mu Epsilon* meeting which was held in Chicago in connection with the meeting of the National Council of Teachers of Mathematics.

Twelve new members were initiated at a dinner meeting in May.

Officers for 1949–50 were: President, Dorothy Karner; Vice-President, Rosemary White; Secretary, Mary Hund; Treasurer, Mary Kilkelly; Corresponding Secretary and Sponsor, Sister Mary Felice.

Joan Bakle was elected President for the year 1950–51 and Sister Mary Petronia has taken over the office of Corresponding Secretary.

Mathematics Club, Harvard University

The following is a summary of the activities of the Harvard *Mathematics Club* for 1949–50. The talks presented were:

The Banach-Tarski paradox, by Prof. Lynn Loomis

Infinite-dimensional representations of the Lorentz group, by Dr. Harish-Chandra

A probability model for simple learning, by Prof. F. Mosteller

A miniature theory, by Prof. D. V. Widder

A functional equation, by Prof. G. deRham

Integral geometry, by Prof. L. Ahlfors

The four-color problem, by Mr. Reese Prosser

Communication and code, by Mr. Ariel Zemach

The odd number six, by Prof. W. V. D. Hodge.

The *Club* participated in the meetings of the Intercollegiate Mathematics Conference of the Boston Area, and sponsored a picnic for faculty and students.

The officers elected for 1950–51 are: President, W. Turanski; Vice-President, A. Zemach; Secretary, H. Gonshor; Treasurer, H. Royden.

The Rogers prizes for student speakers at club meetings were awarded jointly to Mr. Prosser and Mr. Zemach.

Kappa Mu Epsilon, Montclair State Teachers College

Included in the regular monthly meetings of the *New Jersey Beta* chapter of *Kappa Mu Epsilon* were the following programs:

The algebra of logic, by Werner Schanzenbach

Appreciation of elementary mathematics, by Prof. Howard Fehr of Columbia University Teacher's College

Polyhedrons, by Ramon Steinen

Trisection of an angle by means of various curves, by Audrey Jensen

Applications of mathematics to electricity, by Prof. C. Sensale.

The first meeting of the year was an initiation meeting held at the home of Dr. V. S. Mallory at which five members were taken into the organization.

At most of the meetings, members of the organization spoke more fully on topics which they had presented for their initiation. Members of the faculty also presented informal talks on such subjects as hints for the teaching of mathematics. The more formal talks outlined above completed the regular meetings of the chapter.

Members of the *New Jersey Alpha* chapter of *Kappa Mu Epsilon* at Upsala College as well as the mathematics majors and minors from Montclair heard the inspiring talk by Dr. Fehr.

The social program for the year consisted of a Christmas Party held with *Sigma Phi Mu*, the mathematics club; a picnic in May; and finally the annual Spring Banquet held as a farewell to the seniors and to welcome returning alumni. At this banquet copies of Archibald's *History of Mathematics* were awarded to Jack King, Charles Paglieri, and Ramon Steinen, three seniors

having the highest average in mathematics this year.

Officers for 1950–51 are: President, Robert Lynch; Vice-President, Werner Schanzenbach; Secretary, Catherine Buce; Treasurer, Leonard Nichols.

Pi Mu Epsilon, University of Alabama

The *Alabama Alpha* chapter of *Pi Mu Epsilon* held five program meetings during the academic year, 1949–50, in addition to the annual business meeting. Two social functions were held, a Christmas Party and a Spring picnic, on the occasions of the semi-annual initiation ceremonies.

The following papers were presented:

Lattice theory, by Haskell Cohen

New derivations of Taylor's expansion, by Prof. C. L. Seebeck

A method of graphical integration, by Jack Cothren

Mathematics and the sciences, by Dr. A. S. Householder of the Oak Ridge Laboratories.

At one of the meetings the program consisted in discussions of certain problems in this MONTHLY and the *Pi Mu Epsilon Journal*, led by Dr. J. C. Eaves and Mr. Ben Green.

For the second consecutive year, the chapter sponsored a campus-wide competitive examination, designed this year to test the reasoning powers of the competitors rather than their mastery of techniques. Prize winners were William McKee, first prize, Robert Asquith and Donald Swenson, tied for second prize, Wallace Driggers and Walter B. Mitchell.

The following officers were elected for the academic year, 1950–51: Director, Susie Lee Ward; Vice-Director, Dr. Ferdinand Mitchell; Secretary, Mrs. A. M. Jones; Treasurer, Hasell Palmer; Librarian, Dr. J. D. Mancill; Publicity Chairman, Kathleen Cannon; Social Chairman, Ella Jones.

Mathematics Club, University of Buffalo

The activities of the *Mathematics Club* of the University of Buffalo included a business meeting, a picnic, and the following program meetings:

The mathematician in optics, by Dr. Wilkins

The application of mathematics and statistics to the analysis of the economic cycle, by Dr. Zennon Szatrowski, of the Statistics Department

Phyllotaxis, by Dr. Robert Gordon

Intrinsic equations, by Norman Severo

The nomograph, by Henry Hollwedel

Elementary modular arithmetic, by Phyllis Schwartz

Mathematic puzzles and games, by William Schulze

The mathematician in the industrial world, by Dr. Welmers, of the Bell Aircraft Corporation.

The officers elected in April are: President, Paul Schillo; Vice-President, Harlan Stevens; Secretary, Phyllis Schwartz; Treasurer, Bernadine Lippert; Refreshments Chairmen, Norma Wilson and Lois Hunt.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

FOURTH SYMPOSIUM ON APPLIED MATHEMATICS

The Fourth Symposium on Applied Mathematics of the American Mathematical Society, cosponsored by the University of Maryland and the United States Naval Ordnance Laboratory, will be held at the University of Maryland, College Park, Maryland, and the United States Naval Ordnance Laboratory, White Oak, Maryland, on June 22–23, 1951. Sessions will begin at 10:00 A.M. Friday.

The general topic of the Symposium is Fluid Dynamics. Sessions are planned on Turbulence, Foundations, Compressible and Incompressible Flow. Participation in the program is by invitation of the Committee on Arrangements.

Detailed information concerning the program, registration, rooms, meals and travel will be included in a formal announcement of the Symposium to be sent out in May. Inquiries concerning the Symposium should be addressed to Professor M. H. Martin, Chairman of the Committee on Arrangements, University of Maryland, College Park, Maryland.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1951:

Boston University. May 28 to July 7: Professor Chand, seminar in statistics; Dr. Giever, theory of equations; Dr. Scheid, vector analysis. July 9 to August 18: Professor Sobczyk, seminar in analysis, introduction to modern mathematics, advanced analytic geometry; Professor Mode, introduction to mathematical statistics.

Catholic University of America. July 2 to August 11: Dr. Moller, theory of equations; Professor Finan, theory of numbers; Professor Rice, solid analytic geometry and advanced calculus; Mr. Clark, partial differential equations of mathematical physics; Professor Ramler, college geometry, differential equations and analytic projective geometry.

Cornell University. July 2 to August 11: Professor Hunt, projective geometry; Professor Fuchs, teachers' course.

Duke University. June 12 to July 21: Professor Thomas, solid geometry and spherical trigonometry from an advanced standpoint, abstract algebra I; Professor Gergen, algebra from an advanced standpoint, integral equations; Associate Professor Dressel, thesis seminar. July 21 to August 31: Professor Carlitz, abstract algebra II, probability. August 7 to 17: Director W. W. Rankin, Institute for Teachers of Mathematics.

Northwestern University. June 22 to August 25: Differential equations; fundamental concepts of analysis; definite integrals; theory of equations; analytic geometry; theory of statistics; introduction to the theory of numbers; geometry for teachers; functions of a complex variable; topics in linear operators; introduction to the theory of groups; vector analysis; seminar; independent study; thesis; teaching of general mathematics (six-week session only); workshop course in multisensory aids in the teaching of mathematics (last three weeks).

Ohio State University. June 19 to August 31: Professor Rechard, advanced calculus; Professor Mickle, introduction to the theory of complex variables, differential geometry; Professor Miller, projective geometry, fundamental ideas in algebra and geometry; Professor Lazar, history of mathematics.

State University of Iowa. June 12 to August 8: Professor Chittenden, differential equations, theory of manifolds; Professor Wylie, astronomy; Professor Woods, theory of equations, constructive geometry; Professor Price, supervision of mathematics; Professor Muhly, analytic trigonometry, advanced calculus; Professor Hogg, introduction to statistics, matrices and determinants; Professor Oberg, elementary theoretical mechanics, numerical integration of ordinary and partial differential equations.

Syracuse University. July 2 to August 10: Introduction to higher algebra I; college plane geometry; introduction to modern mathematics; history of mathematics; analysis of elementary mathematics; teaching high school mathematics; workshop in mathematics education.

University of Buffalo. July 2 to August 11: Professor Gehman, theory of probability; Professor Montague, curves and their properties, higher algebra; Professor Schneckenburger, differential equations, non-Euclidean geometry.

University of California, Los Angeles. June 18 to August 11: functions of a complex variable; partial differential equations. The following courses are offered under sponsorship of the Institute for Numerical Analysis: sampling methods and stochastic processes; difference and differential equations; asymptotic expansions.

University of Chicago. June 26 to September 1: Professor Albert, linear associative algebras, reading and research in algebra and the theory of numbers; Professor Halmos, Boolean algebras, reading and research in measure theory and functional analysis; Professor Kaplansky, ideals in rings, reading and research in algebra; Professor Graves, existence theorems in the calculus of variations, reading and research in analysis; Professor Segal, unbounded operators on Hilbert space, reading and research in abstract analysis; Professor Chern, differentiable manifolds, reading and research in differential geometry; Professor MacLane, reading and research in algebra and topology.

University of Colorado. June 18 to July 20: Professor Roy, mathematical statistics; Dr. Johnson, workshop in mathematics, advanced course in teaching of mathematics; Professor Kendall, analytic projective geometry, history of mathematics; Professor Britton, introduction to modern algebra, functions of a

complex variable (first and second terms); Mr. Hunt, advanced calculus (first and second terms); Mr. Wagner, vector analysis (first and second terms). July 23 to August 24: Professor Jones, theory of numbers, analytic projective geometry.

University of Detroit. June 25 to August 3: Professor Smith, theory of groups and theory of higher plane curves; Professor Markle, seminar in mathematics; Professor McCarthy, differential equations; Professor Mehlenbacher, theory of functions of a complex variable.

University of Kansas. June 7 to August 4: Professor Bell, projective geometry, matrix and tensor calculus; Mr. Bradt, elementary statistics; Professor Scott, set theory; Professor Wolontis, Fourier series and boundary value problems.

University of Kentucky. June 18 to August 11: Professor Davis, intermediate calculus; Professor Goodman, theory of equations; Professor Downing, advanced calculus (second part); Professor Pence, college geometry; Professor South, calculus of finite differences; Professor Ward, introduction to theory of numbers; Professor Cowling, integral equations.

University of Maryland. June 23 to August 3: Professor Good, higher algebra; Professor Jackson, introduction to projective geometry.

University of Michigan. June 25 to August 17: Professor Artin, arithmetic of rings; Professor Bartels, modern operational mathematics, Fourier series; Professor Carver, mathematical statistics II, finite differences; Professor Craig, theory of estimation; Professor Copeland, theory of probability, mechanics; Professor Darling, significance tests, theory of statistics I; Professor Dwyer, multivariate analysis, computational methods; Professor Hay, vector analysis, theory of plates and shells; Professor Jones, history of algebra, teaching of collegiate mathematics; Professor Kaplan, functions of complex variable with applications, non-linear differential equations; Dr. Leisenring, higher geometry for teachers; Professor Nesbitt, intermediate mathematics of life insurance; Professor Rothe, methods of partial differential equations, linear integral equations; Professor Samelson, functions of a real variable, integral geometry; Professor Thrall, Galois theory, analytic projective geometry; Professor Wilder, foundations of mathematics, general spaces.

University of Minnesota, Department of Mathematics. June 18 to July 28: Professor Hatfield, mathematical recreations, vector analysis; Professor Carlson, theory of geometric constructions, solid analytic geometry; Professor Cameron, probability, calculus of variations. July 30 to September 1: Professor Olmsted, theory of equations; Professor Nering, foundations of geometry, mathematical theory of games of strategy; Professor Kalisch, the real number system; Professor Loud, the mathematics of small vibrations, Laplace transforms.

University of Minnesota, Institute of Technology. June 18 to July 27: Professor Koehler, intermediate and advanced calculus; Professor Munro, vector analysis and theory of complex variable. July 30 to August 31: Professor Warschawski, advanced calculus; Professor Turrittin, vector analysis, dyadics

with applications.

University of Nebraska. June 5 to July 27: Professor Camp, differential equations; Professor Leavitt, vector analysis; Professor Ribeiro, theory of equations; Professor Basoco, differential geometry.

University of North Carolina. June 11 to July 19: Professor Whyburn, foundations of geometry; Professor Cameron, introduction to modern algebra; Professor Hill, elementary mathematical statistics; Professor Garner, calculus of finite differences; Professor Brauer, some recent results in algebra. July 20 to August 28: Professor Lasley, synthetic projective geometry; Professor Linker, differential equations; Professor Jones, general topology.

University of Oklahoma. June 8 to August 3: Mr. LaFon, elementary differential equations, theory of equations; Professor Hassler, fundamental concepts and methods (teachers' course); Dr. Huff, ordinary and partial differential equations; Professor Brixey, higher algebra; Dr. Bernhart, analytic mechanics; Professor Springer, metric differential geometry; Professor Goffman, integral equations.

University of Virginia. June 25 to August 18: Professor Floyd, advanced calculus and applied mathematics; Professor McShane, differential equations and applied mathematics, advanced analysis; Professor Whyburn, transformation theory.

University of Wisconsin. June 25 to August 17: Professor Bing, elementary plane topology; Professor Bruck, survey of the foundations of algebra, rings and fields; Professor Buck, determinants and matrices; Professor Eberlein, advanced calculus; Professor Kleene, higher mathematics for engineers, introduction to the theory of probability; Professor Mayor, college geometry; Professor Schaeffer, higher analysis, potential theory; Professor Young, higher mathematics for engineers, calculus of variations.

University of Wyoming. June 18 to July 20: Professor Steen, theory of equations; Professor S. R. Smith, differential equations, Fourier series; Professor Calvert, projective geometry; Professor Barr, seminar in geometry. July 23 to August 24: Professor Varineau, theory of numbers, fundamental concepts of mathematics; Professor W. N. Smith, partial differential equations, mathematical theory of probability; Professor Schwid, seminar in analysis, college geometry; Professor Neubauer, history of mathematics.

West Virginia University. June 6 to July 17: Professor Cunningham, modern geometry, higher plane curves; Professor Peters, linear algebra; Professor Vehse, advanced calculus. July 18 to August 24: Professor Vest, theory of equations, advanced differential equations; Professor Stewart, advanced calculus, higher plane curves.

PERSONAL ITEMS

Professor C. H. Gingrich of Carleton College has been elected to honorary membership in the American Association of Variable Star Observers.

Mr. J. H. Johnston of Orkney, Scotland has received the Annual Award of

the Duodecimal Society of America.

Assistant Professor Edith R. Schneckenburger, University of Buffalo, was the representative of the Association at the inauguration of Chancellor T. R. McConnell of the University of Buffalo on January 5–6, 1951.

Dr. W. J. Youden of the National Bureau of Standards has been elected to Fellowship in the New York Academy of Sciences.

Carleton College announces: Professor C. H. Gingrich is the Chairman of the Department of Astronomy and Mathematics; Associate Professor K. W. Wegner has been appointed Registrar but is continuing to teach part-time in the Department.

At Eastern Illinois State College: Dr. D. J. Davis, formerly of the Michigan Children's Institute, has been appointed to an assistant professorship; Instructor L. R. Van Deventer has been promoted to an assistant professorship.

Los Angeles City College reports that the Mathematics Department has sponsored the following series of lectures for mathematics teachers of the Los Angeles School System: Theory of Games by Professor H. F. Bohnenblust, California Institute of Technology; Difference Equations by Professor J. W. Green, University of California at Los Angeles; Series and Sums by Professor Hugh Hamilton, Pomona College; Topics in Applied Mathematics by Professor Tobias Dantzig; Recent Developments in Geometric Paper Folding by C. W. Trigg of Los Angeles City College. Also the College sponsored a lecture, Man and Mathematics, by Professor H. W. Turnbull of the University of St. Andrews, Scotland.

Oklahoma Agricultural and Mechanical College announces the following: Dr. A. D. Ziebur, previously graduate assistant at the University of Wisconsin, has been appointed to an assistant professorship; Mr. W. J. Nemerever, formerly an AEC Fellow at the University of Michigan, has been appointed to an instructorship; Lecturer F. F. Bonsall of King's College, Durham University, England, has been appointed Visiting Associate Professor; Mrs. Gillian Ronsall of Dame Allen's School, England, has been appointed Visiting Assistant Professor; Associate Professor H. W. Smith has been promoted to a professorship; Professor O. H. Hamilton is on leave and is at King's College, Durham University, England; Associate Professor Bee Chrystal has retired with the title of Associate Professor Emeritus.

Suffolk University announces the following appointments: Professor N. J. Anderson of Evansville College to the position of Professor of Chemistry and Mathematics; Mr. Harvey Blend, graduate student at the University of Texas, as an Assistant Professor of Mathematics and Physics.

At the University of British Columbia: Assistant Professor F. M. C. Goodspeed of Queens University has been appointed to an associate professorship; Dr. S. W. Nash has been appointed to an assistant professorship; Professor H. A. Heilbronn of the University of Bristol, England, was Visiting Professor during the first semester of the academic year 1950–51.

University of Colorado announces the following appointments: Professor

Emeritus A. J. Kempner as Visiting Lecturer for the year 1950–51; Professor S. N. Roy of the University of North Carolina and Dr. D. A. Johnston of the University of Minnesota as members of the staff of the Summer Session.

University of Delaware reports the following: Dr. E. V. Lewis, associate chemist with the DuPont Company, has been appointed to an assistant professorship; Instructor A. C. Nelson, Jr., has left the University and is now a graduate student at the University of North Carolina.

University of Detroit makes the following announcements: Assistant Professor Emily C. Pixley has been promoted to an associate professorship; Mr. E. R. Lancaster, graduate student engineer with the Chrysler Corporation, has been appointed to an instructorship.

University of Kentucky announces: Dr. M. S. Davis, previously fellow at Yale University, and Dr. J. C. Flack have been appointed to instructorships; Mr. L. F. Boron of the University of Maine and Mr. R. C. Brown, Jr., West Virginia University, have been appointed to part-time instructorships; Mrs. Carolyn S. Leo, Mr. R. H. Sprague, formerly graduate assistant at Ohio State University, Mr. L. O. Thompson of the University of Detroit and Mr. W. M. Zaring have been appointed to graduate assistantships.

At the University of Massachusetts: Mr. Edward Halpern has been appointed to an instructorship; Instructor P. D. Ritger is on leave of absence and is engaged in graduate study at New York University.

University of Oregon reports: Mr. I. J. Christopher, formerly graduate assistant, Mr. K. G. Clemens of Willamette University, Mr. J. E. Maxfield, previously graduate assistant, Mr. O. S. Rothaus, graduate student at Princeton University, and Mr. J. H. Skelton of Southwest Missouri State College have been appointed to instructorships; Instructor F. H. Young is now an AEC Fellow at the University.

University of Pennsylvania announces the following: Professor L. J. Mordell of St. John's College, Cambridge University, was Visiting Professor during the first semester of the current academic year; Professor F. W. Beal has retired with the title of Professor Emeritus.

University of Wichita makes the following announcements: Assistant Professor E. B. Wedel has been promoted to an associate professorship; Miss Sabrina Morlan of Kansas State College has been appointed to an instructorship.

Wellesley College announces the following: Assistant Professor J. W. Warwick of the Department of Astronomy is a part-time member of the Mathematics Department; Mrs. Mary L. Boas has resigned her position as Lecturer.

Mr. A. H. Albert, formerly teaching fellow at the University of Michigan, has been appointed to an instructorship at Kansas State Teachers College.

Assistant Professor K. J. Arrow of Stanford University has been promoted to an associate professorship.

Dr. George Copp of the University of Texas has been appointed to an assistant professorship at North Texas State College.

Assistant Professor W. H. Fagerstrom of City College of New York has been promoted to an associate professorship.

Associate Professor B. E. Gatewood, USAF Institute of Technology, has been promoted to the position of Professor and Head of the Department of Mechanics.

Mr. Martin Goland has been appointed Editor of *Applied Mechanics Reviews*. The editorial office of this journal is located now at Midwest Research Institute, Kansas City, Missouri.

Mr. L. A. Jehn, formerly teaching fellow at the University of Michigan, has been appointed to an assistant professorship at the University of Dayton.

Mr. J. E. Lesch, graduate assistant at Purdue University, has been appointed to an instructorship at Cornell College.

Miss Betty McKnight of Centenary College, Louisiana, has been promoted to an assistant professorship.

Assistant Professor N. S. Mendelsohn of the University of Manitoba has been promoted to an associate professorship.

Assistant Professor Ruth E. O'Donnell of Duquesne University has been promoted to an associate professorship.

Associate Professor C. D. Olds, San Jose State College, has been promoted to a professorship.

Assistant Professor L. L. Rauch of the Department of Aeronautical Engineering, University of Michigan, has been promoted to an associate professorship.

Associate Professor O. E. Stanaitis of the University of Vilna has been appointed to an instructorship at St. Olaf College.

Professor L. W. Swanson of Coe College has accepted a position as Mathematician in the Applied Science Department of International Business Machines Corporation, Chicago, Illinois.

Associate Professor J. F. Wardwell of Colgate University has been promoted to a professorship.

Mr. G. P. Weeg has been appointed to an instructorship at St. Ambrose College.

Instructor William Wells of State Teachers College, Mankato, Minnesota, has been promoted to an assistant professorship.

Dr. D. M. Young, Jr., has been appointed to an instructorship at Harvard University.

Professor Emeritus Lennie P. Copeland of Wellesley College died on January 11, 1951. She was a charter member of the Association.

Assistant Professor L. C. Dawson of Colorado Agricultural and Mechanical College died on May 29, 1950.

Emeritus Assistant Professor L. C. Knight of the College of Wooster died on December 24, 1950. He had been a member of the Association for thirty years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE NEW EDITOR-IN-CHIEF

At a recent meeting of the Board of Governors of the Association, Professor C. B. Allendoerfer of Haverford College was elected Editor-in-Chief of the MONTHLY for a five-year period beginning January 1, 1952. Professor Allendoerfer has been an Associate Editor of the MONTHLY since 1947 and has been in charge of the Department of Classroom Notes since that time. He is now a member of the Board of Governors and has been a Vice-President of the Association. He brings to his new position a wide experience in the affairs of the Association.

After August 1, 1951, articles intended for publication in the MONTHLY should be sent to Professor Allendoerfer at the address: Haverford College, Haverford, Pennsylvania.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following seventy-nine persons have been elected to membership by the Board of Governors on applications duly certified.

C. B. ABLETT, M.S. (Southern Illinois) Asst. Professor, Lebanon Valley College, Annville, Pa.

MRS. EDITH W. AINSWORTH, M.A. (California) Tuscaloosa, Ala.

DOROTHY J. ANDERSON, B.A. (Macalester) Instr., Macalester College, St. Paul, Minn.

FLORENCE R. ANDERSON, M.A. (U.C.L.A.) Research Laboratory Analyst, Northrop Aircraft, Hawthorne, Calif.

J. M. ANDERSON, Student, South Dakota School of Mines and Technology, Rapid City, S. D.

C. B. BAXTER, (Kansas State T. C.) Partner, C. B. Baxter and Company, Kansas City, Mo.

WILLIAM BECK, Student, Kent State University, Ohio.

E. L. BROOKS, B.S. (Alabama Poly.) Biloxi, Miss.

ARTHA J. BURINGTON, M.E. (Maryland) Teacher, Mount Vernon High School, Alexandria, Va.

M. O. BURRELL, B.A. (Emory) Grad. Student, Emory University, Ga.

C. M. CALLAHAN, B.S. (Lincoln Memorial) Teaching Fellow, Alabama Polytechnic Institute, Auburn, Ala.

S. H. CHASEN, B.C.E. (Georgia Tech.) Grad. Student, Emory University, Gla.

R. H. CHU, Student, Wayne University, Detroit, Mich.

J. W. CREELY, M.S. (Pennsylvania) Research Chemist, American Cyanamid Company, Bound Brook, N. J.

A. C. DAUGHTRY, Student, Alabama Polytechnic Institute, Auburn, Ala.

B. C. DELOACH, JR., Student, Alabama Polytechnic Institute, Auburn, Ala.

G. M. EDDINGTON, M.A. (George Peabody) Asso. Professor, Tusculum College, Greeneville, Tenn.

KARL EIDE, Student, Emmanuel Missionary College, Berrien Springs, Mich.

MARY T. FINIGAN, Student, Regis College, Weston, Mass.

A. D. FLESHLER, M.S. (N.Y.U.) Asst. Professor, Champlain College, Plattsburg, N. Y.

A. J. FLYNN, B.S. (Illinois State Normal) Grad. Student, Illinois State Normal University, Normal, Ill.

L. C. GRAUE, Ph.D. (Indiana) Instr., Sacramento State College, Calif.

EDWARD HALPERN, M.A. (Columbia) Instr., University of Massachusetts, Amherst, Mass.

F. C. HATFIELD, M.S. (Minnesota) Instr., Bemidji State Teachers College, Minn.

G. P. HENDERSON, Ph.D. (Toronto) Lecturer, University of Western Ontario, London, Ont.

JESSIE M. HOAG, M.A. (Alabama) Asst. Professor, Southwestern Louisiana Institute, Lafayette, La.

F. X. HOLZHAUER, B.S. (Detroit) Teaching Fellow, University of Detroit, Mich.

J. G. HORNE, JR., M.S. (Tulane) Grad. Fellow, Tulane University, New Orleans, La.

GRACE M. HYDER, B.S. (Massachusetts) Bookkeeper, Bay State Merchants National Bank, Lawrence, Mass.

JANE C. INGERSOLL, M.A. (Michigan) Research Assistant, Los Alamos Scientific Laboratory, N. M.

J. M. IVANOFF, Student, Gonzaga University, Spokane, Wash.

MRS. RUTH KISSEL, M.A. (Houston) Teacher, University of Houston, Tex.

A. R. KNEER, Student, Gonzaga University, Spokane, Wash.

R. C. KROEGER, B.A. (Buffalo) Teaching Fellow, University of Buffalo, N. Y.

MRS. MERRY M. KRUSE, B.S. (Iowa S. C.) Mathematics Analyst, Sandia Corporation, Albuquerque, N. M.

PAOLO LANZANO, Ph.D. (Rome) Instr., St. Louis University, Mo.

D. B. LARSON, A.B. (Houghton) Grad. Student, University of Buffalo, N. Y.

ROGER LESSARD, B.A.Sc. (Montreal) Asst. Professor, Ecole Polytechnique, Montreal, Que.

W. W. LEUTERT, Dr.Sc.Math. (Zurich) Asst. Professor, University of Maryland, College Park, Md.

F. H. LLOYD, B.S. (Westminster) Asst. Instructor, University of Missouri, Columbia, Mo.

LEE LORCH, Ph.D. (Cincinnati) Asso. Professor, Fisk University, Nashville, Tenn.

J. L. MADDOX, JR., B.E.E. (Alabama Poly.) Teaching Fellow, Albama Polytechnic Institute, Auburn, Ala.

F. J. MALAK, B.S. (Ohio State) Instr., Youngstown College, Ohio.

P. J. MCCARTHY, B.S. in E.E. (Notre Dame) Teaching Fellow, University of Notre Dame, Ind.

W. C. MCCLUNG, Student, College of Wooster, Ohio.

EUGENE MCGILLICUDDY, B.S. (Worcester Poly.) Instr., College of the Holy Cross, Worcester, Mass.

MRS. ANITA B. MILAM, A.B. (Washington Missionary) Instr., Washington Missionary College, Takoma Park, Md.

H. S. MOREDOCK, JR., Ph.D. (California) Asst. Professor, Sacramento State College, Calif.

PRASERT NA NAGARA, B.Sc.Agr. (Philippines) Registrar, College of Agriculture, Bangkhen, Bangkok, Thailand.

FRANCES A. NORTON, A.B. (Birmingham Southern) Teaching Fellow, Alabama Polytechnic Institute, Auburn, Ala.

DANIEL ORLOFF, S.M. (Chicago) Instr., Southern Illinois University, Carbondale, Ill.

MRS. ERNA H. PEARSON, M.A. (Texas) Atlanta, Ga.

G. M. PETERSEN, M.A. (Stanford) Grad. Student, University of Toronto, Ont.

O. A. PLAIT, B.S. (Illinois Tech.) U. S. Navy, Chicago, Ill.

J. E. PRYOR, Ph.D. (Louisiana State) Professor, Harding College, Searcy, Ark.

J. W. RICHARDSON, Student, South Dakota School of Mines and Technology, Rapid City, S. D.

J. R. RICUPITO, Student, University of Florida, Gainesville, Fla.

N. W. RIEBE, B.S. (New Mexico) Research Assistant, University of California, Los Alamos, N. M.

J. F. RIES, M.A. (Delaware) Instr., Kings College, Del.

B. D. RUDIN, B.S. (C.I.T.) Grad. Assistant, University of Southern California, Los Angeles, Calif.

W. A. RUTLEDGE, Ph.D. (Tennessee) Asst. Professor, Alabama Polytechnic Institute, Auburn, Ala.

J. M. SANDY, B.S. in Ed. (Missouri) Asst. Instructor, University of Missouri, Columbia, Mo.

P. T. SCHAEFER, Student, University of Rochester, N. Y.

E. J. SCHWEPPE, B.S. (Missouri Valley) Grad. Student, University of Illinois, Urbana, Ill.

ESTHER SEIDEN, Ph.D. (California) Asst. Professor, University of Buffalo, N. Y.

MAE DELL SENTELL, M.S. (Alabama Poly.)

Instr., Alabama Polytechnic Institute, Auburn, Ala.

S. W. SHELTON, JR., M.A. (George Peabody) Asso. Professor, Northwestern State College, Natchitoches, La.

J. A. SILVA, M.A. (Duke) Instr., Duke University, Durham, N. C.

SISTER MARY SYLVESTER, M.S. (St. Louis) Instr., Mundelein College, Chicago, Ill.

W. I. STEINKAMP, Student, Gonzaga University, Spokane, Wash.

T. R. STENGLE, Student, Franklin and Marshall College, Lancaster, Pa.

W. J. STRAUSS, M.S. (Chicago) Chicago, Ill.

W. G. THORNTON, M.S. (Florida State) Teacher, Madison High School, Fla.

C. H. TROSS, B.S. (Illinois) Mathematician, U. S. Naval Proving Ground, Dahlgren, Va.

T. E. VICHICH, B.S. (Michigan Tech.) Asst. Professor, Michigan College of Mining and Technology, Houghton, Mich.

G. E. WALDMAN, M.A. (Columbia) Teacher, Tottenville High School, Staten Island, N. Y.

C. R. WALKER, M. in E.E. (Brooklyn Poly.) Engineer, American Telephone and Telegraph Company, New York, N. Y.

ANDREW WEISS, Student, University of Detroit, Mich.

EDMUND WESOLOWSKI, Student, Elmhurst College, Ill.

REPORT OF THE TREASURER FOR THE YEAR 1950

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1950. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

I. TOTAL FUNDS OF THE ASSOCIATION ON JANUARY 1, 1950

M & T Trust Co., Buffalo	$ 7,926.08	Current Fund	$ 7,926.08
Securities	63,600.63	Carus Fund	10,337.74
		Chace Fund	9,003.16
		Houck Fund	9,861.11
		Chauvenet Fund	653.95
		General Fund	33,744.67
	$71,526.71		$71,526.71

II. CURRENT FUND

Balance, January 1, 1950	$ 7,926.08	MONTHLY	
Dues	16,830.03	Publication	$12,750.66
Initiation fees	890.00	Reprints (net)	227.73
Subscriptions	3,618.10	Editor's Office	448.65
Sales of back numbers	1,127.87	Secretary-Treasurer's Office	
Advertisements	3,392.00	Clerical help	5,963.33
Contributions for publication of		Postage	768.10
the MONTHLY	404.00	Printing	538.00
Sale of exchange periodicals	262.75	Office expenses	248.51
Interest on General Fund	1,088.75	Bank fees	109.54
Income from Hardy Fund	120.00	Board of Governors	2,146.50
		Meetings	572.82
		Representatives	122.85
		Subventions	983.39
		List of Officers and Members	1,610.47
		Transfer to General Fund	464.97
		Balance, December 31, 1950	8,705.06

III. Carus Fund

Balance January 1, 1950.....	$10,337.74	Honoraria, Monographs 9, 10...	$ 600.00
Sales of Monographs.....	3,926.96	Printing Monograph 9.....	3,165.93
Interest.....	328.68	Printing Monograph 10.....	3,150.55
Increase in value of securities....	24.98	Balance, December 31, 1950....	7,701.88

IV. Chace Fund

Balance, January 1, 1950.....	$ 9,003.16	Balance December 31, 1950.....	$10,117.07
Sales of Papyrus.....	290.00		
Sale of Slaught Papers.....	514.46		
Interest.....	287.60		
Increase in values of securities..	21.85		

V. Houck Fund

Balance, January 1, 1950.....	$ 9,861.11	Clerical and editorial expense, Index.....	175.00
Interest.....	328.68	Printing Index.....	1,827.60
Sale of Index.....	2.00	Balance, December 31, 1950....	8,214.71
Increase in value of securities....	24.98		

VI. Chauvenet Fund

Balance, January 1, 1950.....	635.95	Award of Chauvenet Prize.....	50.00
Interest.....	20.54	Balance, December 31, 1950....	626.05
Increase in value of securities....	1.56		

VII. General Fund

Balance, January 1, 1950.....	$33,744.07	Contribution to International Congress.....	$ 1,000.00
Increase in value of securities....	82.73		
Transfer from Current Fund....	464.97	Balance, December 31, 1950....	33,292.37

VIII. Total Funds of the Association on December 31, 1950

Current Fund.....	$ 8,705.06	M & T Trust Co., Buffalo.....	$ 8,705.06
Carus Fund.....	7,701.88	Securities.....	59,952.08
Chace Fund.....	10,117.07		
Houck Fund.....	8,214.71		
Chauvenet Fund.....	626.05		
General Fund.....	33,292.37		
	$68,657.14		$68,657.14

MAY MEETING OF THE WISCONSIN SECTION

The eighteenth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Marquette University, Milwaukee, Wisconsin, on Saturday, May 13, 1950. Professor J. R. Mayor, Chairman of the Section, presided at the morning and afternoon sessions.

About one hundred persons were present, including the following forty-seven members of the association: K. J. Arnold, R. H. Bardell, R. D. Bartz, Leon Battig, L. J. Berner, A. C. Berry, Leonard Bristow, R. C. Buck, B. H. Colvin, Rev. L. A. V. DeCleene, W. S. Ericksen, H. P. Evans, E. R. Finkbeiner, Harold

Glander, W. A. Golomski, E. G. Harrell, Martha Hildebrandt, R. T. Hood, C. C. Hsiung, R. C. Huffer, Rev. M. L. Jautz, J. B. Kelly, J. F. Kenney, R. E. Langer, Vivian E. Larson, C. C. MacDuffee, A. E. May, J. R. Mayor, Edward McGaughy, P. E. Meadows, Genevieve S. Meyer, Elli Otteson, L. E. Overn, G. A. Parkinson, H. P. Pettit, Sister M. Mirabella, Sister Mary Felice, Sister Mary Petronia, Elizabeth S. Sokolnikoff, Abraham Spitzbart, J. V. Talacko, J. R. Thompson, R. D. Wagner, N. M. Watermolén, Frances Weisbecker, Louise A. Wolf, Lillian B. Zarling.

At the business meeting held after the morning program, the following officers were elected for the coming year: Chairman, E. G. Harrell, Platteville State Teachers College; Secretary, Louise A. Wolf, University of Wisconsin in Milwaukee; Program Committee Chairman, Rev. L. A. V. DeCleene, St. Norbert's College. The 1951 meeting of the Section will be held at Carroll College.

The program consisted of the following papers:

1. *Arithmetic density*, by Professor R. C. Buck, University of Wisconsin.

On the set of positive integers I, both a finitely additive measure D^*, and a topology may be defined by recourse to the arithmetic progressions $(\lambda n - b)$ as intervals. The problem of the measurability of a set of integers, and its topological properties, lead to new multiplicative and semi-multiplicative functions, and become problems in the theory of numbers. As examples, we have: (i) the set P of primes, of squares, and of cubes, has measure 0; (ii) the set of squares is closed, the set $P \cup \{1\}$ is closed, but there exist polynomials Q such that $Q(I)$ is not closed; (iii) the quasi-progression $([\beta n])$, for irrational β, is everywhere dense in I, and has inner density 0.

2. *On letting a coin decide*, by Professor A. C. Berry, Lawrence College.

3. *The Mathematical Association of America*, by Professor R. E. Langer, University of Wisconsin.

4. *The Wisconsin Mathematics Council*, by Miss Margaret Striegl, Wauwatosa High School, introduced by the Secretary.

This speaker, who represented the Wisconsin Mathematics Council at the Delegate Assembly sponsored by the National Council of Teachers of Mathematics in Chicago, April 13–15, reported on the work of that group. Thirty-nine official delegates were in attendance to discuss problems which large area groups face when organizing to carry on a better program in the field of mathematics. Decisions as to eligibility requirements, group responsibilities, group privileges, and National Council obligations were submitted as recommendations to the board of directors. The members of the board, pleased with the work of the Delegate Assembly, arranged for another meeting at the next annual convention. The Wisconsin Mathematics Council affiliated with the National Council in December, 1949, after having reorganized its own group during the previous year. The membership has increased considerably. The publication, *Wisconsin Teacher of Mathematics*, to be issued four times a year, will help to keep the members informed of current activities, and will give suggestions for teachers of mathematics at all levels.

5. *Mathematics teacher training programs in America*, by Mr. George Bullis, State Teachers College, Platteville, introduced by the Chairman.

This paper gave a summary of results of a questionnaire on the pre-service training of mathematics teachers. The survey was designed to show how inadequately the present programs in our colleges for teacher education conform to the recommendations of the Joint Commission Report of 1940.

6. *An analysis of some issues and problems faced by the secondary teacher of mathematics*, by Miss Martha Hildebrandt. Proviso Township High School, Maywood, Illinois.

After a brief resume of some suggested "new college admissions requirements recommended" and of an administration suggested program for secondary schools, such other issues as college going and non-college going mathematics, teaching for meaning and understanding, isolation of concepts and courses, using teaching aids for developmental purposes, teaching reading in mathematics, guidance and psychological development through mathematics, were discussed.

LOUISE A. WOLF, *Secretary*

CALENDAR OF FUTURE MEETINGS

Joint meeting with American Society for Engineering Education, Michigan State College, East Lansing, June 25–26, 1951.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 5, 1951.

ILLINOIS, University of Illinois, Urbana, May 11–12, 1951.

INDIANA, May 5, 1951.

IOWA, Wartburg College, Waverly, April 20–21, 1951.

KANSAS, University of Kansas, Lawrence, April 7, 1951.

KENTUCKY, Eastern Kentucky State College, Richmond, April 28, 1951.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, United States Naval Academy, Annapolis, Maryland, April 28, 1951.

METROPOLITAN NEW YORK, Manhattan College, April 7, 1951.

MICHIGAN

MINNESOTA, College of St. Benedict, St. Joseph, April 28, 1951.

MISSOURI, Central College, Fayette, April 6, 1951.

NEBRASKA, University of Nebraska, Lincoln, April 21, 1951.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 21, 1951.

OKLAHOMA

PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 24, 1951.

ROCKY MOUNTAIN, Colorado State College of Education, Greeley, April 20–21, 1951.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN

TEXAS, Southern Methodist University, Dallas, April 27–28, 1951.

UPPER NEW YORK STATE, Hamilton College, Clinton, May 5, 1951.

WISCONSIN, Carroll College, Waukesha, May 12, 1951.

Prentice-Hall
COLLEGE TEXTS

COLLEGE MATHEMATICS—A First Course

Second Edition—1951

By *W. W. Elliott, Duke University,* and *Edward R. C. Miles, The Johns Hopkins University*

Outstanding in the new revision of this long-popular text is the large number of *completely fresh* problems—over 1200 of them. Drawing on their recent experience in both the theoretical and practical fields, the authors have given these problems a practicality, vigor, and immediacy of application that add greatly to the interest of the course.

Another improvement in this edition is the entirely new chapter on "Inequalities." Here the authors carefully develop both graphical and practical solutions.

The arrangement of the book, which helped make the first edition so popular, has been retained. As before, the authors offer a carefully planned, expertly written, non-integrated treatment of the topics suitable to a freshman course. Algebra, trigonometry, and introductory analytic geometry and calculus are each taken up in turn. This arrangement helps students to follow each part of the course and gain a clearer understanding of both principles and applications.

Published June 1951 About 472 pages 6 x 9 inches

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Business students with the barest minimum of training in mathematics are provided with a mathematics background needed for successful study of finance, commerce, accountancy, business statistics, and related subjects. The text, suitable for a one-semester or full-year course, has been specifically designed for use in schools and colleges of business administration, commerce, finance, and in junior colleges.

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PRESENTING the first fifteen chapters of Brink's *College Algebra,* Second Edition, this text supplies the needs of college students who have had only one year of high-school algebra and desire to prepare for strictly college mathematics. *To be published in June.*

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AFTER two new introductory chapters, this book presents in full Chapters X through XXVI of *College Algebra,* Second Edition. It supplies the needs of college students who do not require a review of high-school algebra. *To be published in September.*

Coming in May

THE MATHEMATICS OF FINANCE

By FRANKLIN C. SMITH, College of St. Thomas

OFFERING an introductory course in the mathematics of finance to the student of business administration, this text emphasizes basic principles rather than special formulas. An appendix of mathematical tables, a review of algebraic topics, and exercises are included. *To be published in May.*

THE APPLETON-CENTURY MATHEMATICS SERIES

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New Spring Publications

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H. L. RIETZ and A. R. CRATHORNE, ***late of the University of Illinois***
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Professor Peters has kept in mind the objectives and methods of the original authors while incorporating new material drawn from his own wide experience. A more extended review of the topics of secondary school algebra is given. Chapter I now begins with a discussion of the number system and the elementary operations of algebra, Chapters II through VI constitute a review of the rational processes of elementary algebra. New topics, such as functional notation, graphs, and determinants have been introduced, and exercise material has been greatly expanded throughout. *April 378 pages About $3.00*

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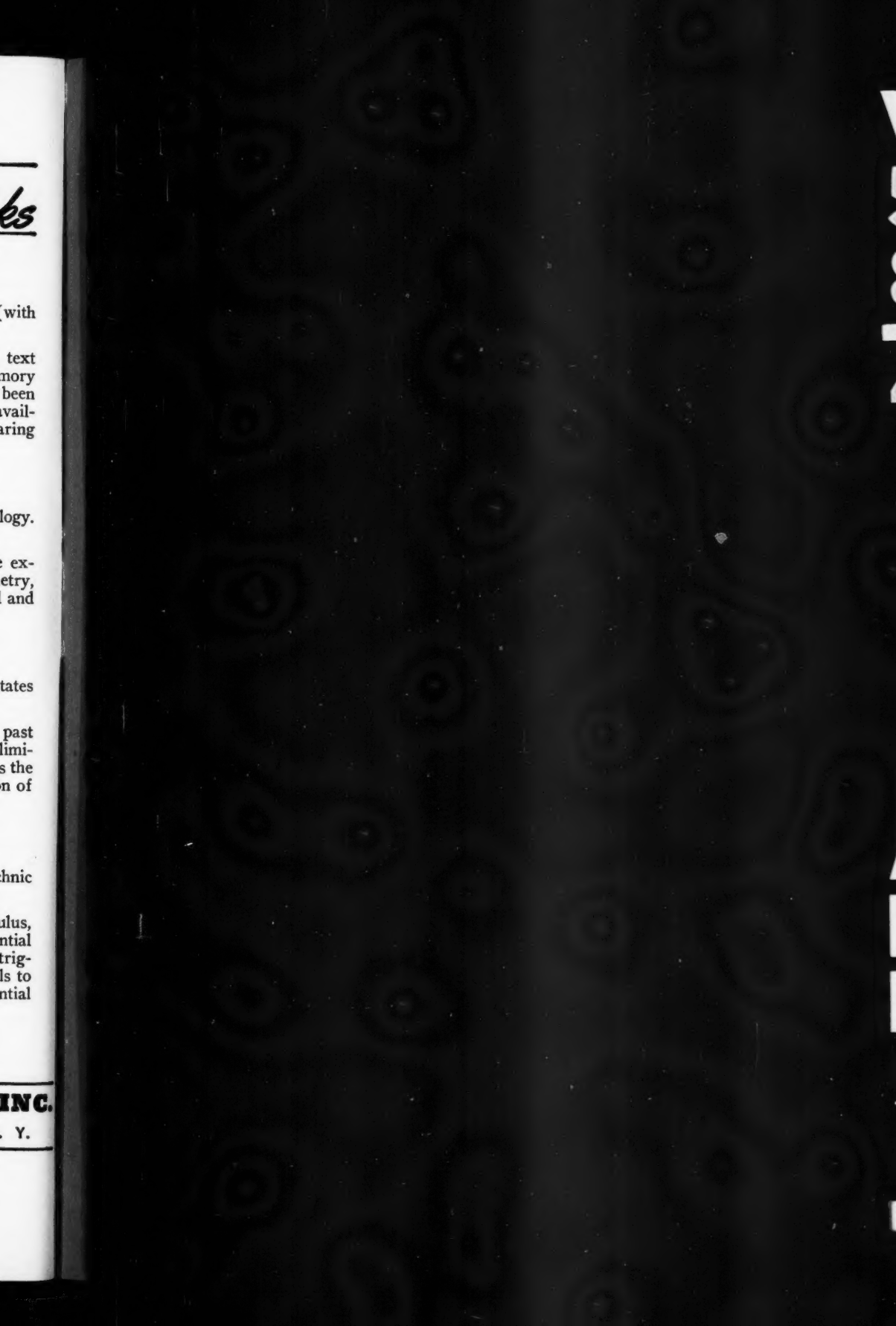

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